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A general theory of self-similarity

Tom Leinster*

Abstract

A little-known and highly economical characterization of the real interval $[0, 1]$, essentially due to Freyd, states that the interval is homeomorphic to two copies of itself glued end to end, and, in a precise sense, is universal as such. Other familiar spaces have similar universal properties; for example, the topological simplices Δ^n may be defined as the universal family of spaces admitting barycentric subdivision. We develop a general theory of such universal characterizations.

This can also be regarded as a categorification of the theory of simultaneous linear equations. We study systems of equations in which the variables represent spaces and each space is equated to a gluing-together of the others. One seeks the universal family of spaces satisfying the equations. We answer all the basic questions about such systems, giving an explicit condition equivalent to the existence of a universal solution, and an explicit construction of it whenever it does exist.

Key words: recursion, self-similarity, final coalgebra, real interval, barycentric subdivision, fractal, categorification, colimit, bimodule, profunctor, flat functor

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Introduction

Ask a mathematician for a definition of the topological space $[0, 1]$, and they will probably define it as a subspace of the real line \mathbb{R} . Pushed for a definition of \mathbb{R} itself, they might, with some reluctance, mention its construction by Dedekind cuts or Cauchy sequences, or its characterization as a complete ordered field. The reluctance stems from the fact that in everyday practice, most mathematicians do not think of real numbers as Dedekind cuts or equivalence classes of Cauchy sequences; and while the characterization as a complete ordered field is better from this point of view, it involves far more structure than is relevant to the mere topology of the interval.

There is, however, a simple characterization of the topological space $I = [0, 1]$ that reflects rather accurately how it is used in topology. Roughly, it says the following: if we define a **path** in a space S to be a (continuous) map $I \longrightarrow S$, then I has exactly the structure needed in order that paths can be composed, and it is universal as such.

Let us make this precise. To speak of composition of paths, we first need to know that every path has a starting point and a finishing point. Whenever we have a pair of paths, the first finishing where the second starts, we wish to be able to compose them to form a new path. These requirements correspond to I coming equipped with two basepoints, 0 and 1, and an endpoint-preserving map to its ‘doubling’—the space obtained by taking two copies of I and gluing the second basepoint of the first to the first basepoint of the second. Moreover, the two basepoints are distinct and, as singleton subsets, closed.

Let \mathcal{D} be the category in which an object is a space equipped with two distinct, closed basepoints and an endpoint-preserving map to its doubling; then we have just observed that I , with some extra structure, is an object of \mathcal{D} . The characterization of I is that it is, in fact, the terminal object. This is a topological version of a theorem of Freyd (2.2). It characterizes (an interval of) the real numbers using only the extremely primitive concepts of continuity and gluing.

Other important spaces have similar characterizations. For example, the embodiment of the concept of convergent sequence is the space $\mathbb{N} \cup \{\infty\}$ (the one-point compactification of the discrete space \mathbb{N}), in the sense that a convergent sequence in an arbitrary space S amounts to a continuous map $\mathbb{N} \cup \{\infty\} \longrightarrow$

S . There is a precise sense in which the pair $(X_1, X_2) = (\{\star\}, \mathbb{N} \cup \{\infty\})$ is the universal solution to the system of ‘equations’

$$X_1 \cong X_1 \tag{1}$$

$$X_2 \cong X_1 + X_2. \tag{2}$$

(Here $\{\star\}$ is the one-point space and $+$ denotes coproduct or disjoint union of spaces.) Indeed, let \mathcal{D} be the category in which an object is a pair (X_1, X_2) of topological spaces together with a pair of maps $(X_1 \longrightarrow X_1, X_2 \longrightarrow X_1 + X_2)$; then the terminal object of \mathcal{D} is $(\{\star\}, \mathbb{N} \cup \{\infty\})$.

Another example characterizes the standard topological simplices Δ^n . Let Δ_{inj} be the category of totally ordered sets $[n] = \{0, \dots, n-1\}$ ($n \geq 0$) and order-preserving injections. There is a functor $I : \Delta_{\text{inj}} \longrightarrow \mathbf{Top}$ assigning to $[n]$ the topological n -simplex Δ^n . This functor I is fundamental: by a stock categorical construction it induces the adjunction $\mathbf{Top} \rightleftarrows \mathbf{Set}^{\Delta_{\text{inj}}^{\text{op}}}$ on which much of algebraic topology is built. (The first functor here is the singular semisimplicial set of a space, and the second is geometric realization.) And I has a universal property similar in character to the two already mentioned: it is the universal functor admitting the combinatorial process of barycentric subdivision (Example 10.12).

The spaces mentioned so far are standard objects of classical algebraic topology, but the same kind of universal characterization also captures some non-classical spaces. For example, there are similar characterizations of certain fractals—spaces that seem to be the epitome of complexity, but turn out to have simple universal properties. Conjecturally (2.11), this includes the Julia set of any complex rational function.

We use the term *self-similarity* in a ‘global’ sense. The interval $[0, 1]$, for example, is called self-similar because it is homeomorphic to a gluing-together of two copies of itself. ‘Local’ statements of self-similarity say something like ‘almost any small pattern observed in one part of the object can be observed throughout the object, at all scales’. (See for instance Chapter 4 of Milnor [Mil], where such statements are made about Julia sets.) Global statements say something like ‘the whole object consists of several smaller copies of itself glued together’; more generally, there may be a whole family of objects, each of which can be described as several objects in the family glued together. The purpose of this paper is to develop a theory of global self-similarity.

Viewed from another angle, this is a theory of recursive decomposition. Our first example concerned a recursive decomposition of the real interval. In the second, the isomorphisms (1) and (2) can be interpreted as a pair of mutually recursive type definitions, in the sense of computer science. Here we only study recursive characterizations of sets and topological spaces. It may be possible to extend the theory to encompass other types of space, hence other types of recursive decomposition or self-similarity: conformal, statistical, type-theoretic, and so on.

Another possibility is to develop the algebraic topology of self-similar spaces, for which the usual homotopical and homological invariants are often useless: in the case of a connected fractal subset of the plane, for example, they only give us π_1 , which is typically either infinite-dimensional or trivial. However, a characterization by a recursive system of equations is a discrete description, and so might lead to useful invariants.

We set up the basic language in §1 and §2. The main aim there is to motivate the definitions of *equational system* and of *universal solution* of an equational system. Informally, an equational system is a system of equations in which each variable, representing a space, is equated to a colimit or gluing-together of the others. A universal solution of such a system is a solution with a particular universal property. For example, the result above states that the real interval (equipped with some extra structure) is the universal solution of a certain simple equational system.

With the language set up, the principal results of the rest of the paper can be summarized (§3). These results completely answer all the basic questions about existence, construction and recognition of the universal solutions of equational systems.

Category theory is essential here, for two reasons. First, our spaces are to be characterized by universal properties. Second, the appropriate general notion of ‘gluing’ is the categorical notion of colimit. Further categorical concepts become needed, and are explained, as the theory develops.

Related work Various other theories are related to this one. Symbolic dynamics [LM] seems most closely related to the case of *discrete* equational systems (§1). Iterated function systems [Fal, Hut] are related, but differ crucially in that they take place inside a fixed ambient space, whereas we are concerned with spaces in the abstract; see Examples 10.10 and 10.11.

The motivating example for this work was the theorem of Freyd on the real interval [Fre2, Fre3]. This in turn was inspired by a theorem of Pavlović and Pratt [PP]. Their results are part of a long line of work on terminal coalgebras in computer science. (See [Fre1], for instance, and [Adá] for a survey.) In that context, (co)recursively defined data types occur as terminal coalgebras; they are a non-topological analogue of our recursively decomposable spaces. Freyd’s Theorem stimulated other related work, in particular that of Escardó and Simpson [ES]. Escardó also obtained a topological version of Freyd’s Theorem [Esc], different from ours.

A paper of Barr has some obvious similarities to the present work [Barr]. He discusses terminal coalgebras for an endofunctor and the metrics associated with them. However, the class of endofunctors that he considers has little overlap with the class considered here. The categories on which his endofunctors act always have a terminal object, and his terminal coalgebras can be constructed as limits; contrast 6.2 below.

Recent work of Karazeris, Matzaris and Velebil [KMV] builds on the work here, giving new theorems, and new proofs of old theorems, in the general theory of categorical coalgebras.

A short survey of the work contained in this paper is available [Lei3].

Notation and terminology The sum (coproduct) of a family $(X_i)_{i \in I}$ of objects of a category is written $\sum_i X_i$. If $X_i = X$ for all i then the sum is written $I \times X$. The sum of a finite family X_1, \dots, X_n of objects is written as $X_1 + \dots + X_n$, or as 0 if $n = 0$.

Given categories \mathcal{A} and \mathcal{B} , the category whose objects are functors from \mathcal{A} to \mathcal{B} and whose morphisms are natural transformations is written $[\mathcal{A}, \mathcal{B}]$.

Top is the category of *all* topological spaces and continuous maps.

A **discrete** category is one in which the only maps are the identities. Small discrete categories are therefore just sets. A **finite** category is one with only finitely many maps (hence only finitely many objects). A category is **connected** if it is nonempty and cannot be written as the coproduct of two nonempty categories.

The set \mathbb{N} of natural numbers is taken to include 0.

The cardinality of a finite set S is denoted $|S|$.

We will use extensively the language of modules (in the sense of category theory), also called bimodules, profunctors or distributors [Bén, Law]. An introduction to modules can be found in Appendix A; here we just state the basic conventions.

Given categories \mathbb{A} and \mathbb{B} , a **module**

$$M : \mathbb{B} \multimap \mathbb{A}$$

is a functor $M : \mathbb{B}^{\text{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$. (When \mathbb{A} and \mathbb{B} are monoids construed as one-object categories, such a module M is a set with a compatible left \mathbb{A} -action and right \mathbb{B} -action.) For objects $a \in \mathbb{A}$ and $b \in \mathbb{B}$, we write

$$m : b \multimap a$$

to mean $m \in M(b, a)$. Thus, a module $M : \mathbb{B} \multimap \mathbb{A}$ is an indexed family $(M(b, a))_{b \in \mathbb{B}, a \in \mathbb{A}}$ of sets together with actions:

$$\begin{array}{ccc} b \multimap a \xrightarrow{f} a' & \text{gives} & b \multimap a' \xrightarrow{fm} a', \\ b' \xrightarrow{g} b \multimap a & \text{gives} & b' \multimap a \xrightarrow{mg} a. \end{array}$$

These are required to satisfy axioms: $(f'f)m = f'(fm)$, $1m = m$, and dually; and $(fm)g = f(mg)$.

A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ can be viewed as a module $\mathbf{1} \multimap \mathbb{A}$, where $\mathbf{1}$ denotes the category with one object and only the identity arrow. In this special case, the ‘ fm ’ notation above becomes the following: given an arrow $f : a \longrightarrow a'$ in \mathbb{A} and an element $x \in X(a)$, we write fx for the element $(Xf)(x) \in X(a')$. Similar notation (‘ yg ’) is used for contravariant functors $Y : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set}$.

We will also use commutative diagrams involving crossed arrows \multimap , as explained in Appendix A.

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1 Discrete equational systems

We work our way up to the concept of equational system by first considering an important special case, discrete equational systems. It illustrates many aspects of the general case, but in a simpler setting.

A discrete equational system can be thought of as a system of linear equations such as

$$x_1 = 2x_1 + 5x_2 + x_3 \quad (3)$$

$$x_2 = x_2 \quad (4)$$

$$x_3 = 4x_1 + x_2. \quad (5)$$

Better, it can be thought of as a *categorification* of such a system: the variables x_i represent spaces, addition is coproduct, and the equalities are really isomorphisms. General equational systems can also be thought of as a categorification of such systems of equations—but a more subtle one.

We introduce discrete equational systems using two examples.

The Cantor set

The **Cantor set** is the topological space $2^{\mathbb{N}^+}$, that is, the product $2 \times 2 \times \dots$ of countably infinitely many copies of the discrete two-point space $2 = \{0, 1\}$. (Here \mathbb{N}^+ is the set $\{1, 2, \dots\}$ of positive integers.) The Cantor set is often regarded as a subset of the real interval $[0, 1]$ via the embedding

$$(m_n)_{n \geq 1} \mapsto \sum_{n \geq 1} 2m_n \cdot 3^{-n}$$

($m_n \in \{0, 1\}$), but here we will only consider it as an abstract topological space.

The Cantor set satisfies an ‘equation’: $2^{\mathbb{N}^+} = 2^{\mathbb{N}^+} + 2^{\mathbb{N}^+}$. More precisely, there is a canonical isomorphism

$$\iota : 2^{\mathbb{N}^+} \xrightarrow{\sim} 2^{\mathbb{N}^+} + 2^{\mathbb{N}^+},$$

where $\iota(0, m_2, m_3, \dots)$ is the element (m_2, m_3, \dots) of the first copy of $2^{\mathbb{N}^+}$, and $\iota(1, m_2, m_3, \dots)$ is the element (m_2, m_3, \dots) of the second copy of $2^{\mathbb{N}^+}$. The pair $(2^{\mathbb{N}^+}, \iota)$ has, moreover, a universal property: it is terminal among all pairs (X, ξ) where X is a topological space and $\xi : X \longrightarrow X + X$ is any (continuous) map. In other words, for any such pair (X, ξ) there is a unique map $\bar{\xi} : X \longrightarrow 2^{\mathbb{N}^+}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\xi} & X + X \\ \bar{\xi} \downarrow & & \downarrow \bar{\xi} + \bar{\xi} \\ 2^{\mathbb{N}^+} & \xrightarrow{\iota} & 2^{\mathbb{N}^+} + 2^{\mathbb{N}^+} \end{array}$$

commutes. This can easily be verified directly; it is also a very special case (Example 10.1) of the theory developed in this paper.

Some terminology will allow us to express this universal property more succinctly.

Definition 1.1 Let \mathcal{C} be a category and G an endofunctor of \mathcal{C} (that is, a functor $\mathcal{C} \rightarrow \mathcal{C}$). A **G -coalgebra** is a pair (X, ξ) where $X \in \mathcal{C}$ and $\xi : X \rightarrow G(X)$. A **map** $(X, \xi) \rightarrow (X', \xi')$ of G -coalgebras is a map $X \rightarrow X'$ in \mathcal{C} such that the evident square commutes.

Example 1.2 Let \mathcal{C} be the category of modules over some commutative ring, and let G be the endofunctor defined by $G(X) = X \otimes X$. Then a G -coalgebra is a (not necessarily coassociative) coalgebra in the algebraists' sense.

Now let \mathcal{C} be the category **Top** of topological spaces, and let G be the endofunctor defined by $G(X) = X + X$. A G -coalgebra is a space X together with a map $\xi : X \rightarrow X + X$. The universal property of the Cantor set is that $(2^{\mathbb{N}^+}, \iota)$ is the **terminal coalgebra**, that is, the terminal object in the category of coalgebras.

In our example, the structure map ι of the terminal coalgebra is an isomorphism. This is not coincidence, as the following elementary result reveals.

Lemma 1.3 (Lambek [Lam]) *Let \mathcal{C} be a category and G an endofunctor of \mathcal{C} . If (I, ι) is terminal in the category of G -coalgebras then $\iota : I \rightarrow G(I)$ is an isomorphism.* \square

A G -coalgebra (X, ξ) in which $\xi : X \rightarrow G(X)$ is an isomorphism is called a **fixed point** of G .

Spaces of walks

We turn now to a different topological object with a different universal property. Consider walks on the natural numbers, of the following type:

- start at some position n
- with each tick of the clock, take one step left or one step right—unless at position 0, in which case stay there
- continue forever.

(One might consider imposing a different rule at 0; see Example 10.4.)

Let W_n be the set of all walks starting at position n . Formally, W_n is the set of elements $(a_0, a_1, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $a_0 = n$ and for all $r \in \mathbb{N}$, *either* $a_r > 0$ and $a_{r+1} \in \{a_r - 1, a_r + 1\}$, *or* $a_r = a_{r+1} = 0$. There is a (profinite) topology on W_n generated by taking, for each $n, a_0, \dots, a_n \in \mathbb{N}$, the set of all walks beginning (a_0, \dots, a_n) to be closed. So we have a family $(W_n)_{n \in \mathbb{N}}$ of spaces, and this is the ‘topological object’ that we will characterize by a universal property.

First note that the spaces W_n satisfy some ‘equations’, or rather, isomorphisms. A walk starting at position $n > 0$ consists of either a step left followed by a walk starting at $n - 1$, or a step right followed by a walk starting at $n + 1$. Thus, there is a canonical isomorphism

$$\iota_n : W_n \xrightarrow{\sim} W_{n-1} + W_{n+1}$$

for each $n > 0$. Similarly, a walk starting at position 0 consists of a null step followed by another walk starting at 0, so there is a canonical isomorphism

$$\iota_0 : W_0 \xrightarrow{\sim} W_0.$$

(In fact, W_0 is the one-point space, so ι_0 is the identity.)

These isomorphisms can be expressed as follows. The family $W = (W_n)_{n \in \mathbb{N}}$ is an object of the category $\mathcal{C} = \mathbf{Top}^{\mathbb{N}}$ of sequences of spaces. There is an endofunctor G of \mathcal{C} defined by

$$(G(X))_n = \begin{cases} X_{n-1} + X_{n+1} & \text{if } n > 0 \\ X_0 & \text{if } n = 0 \end{cases} \quad (6)$$

($X \in \mathcal{C}$, $n \in \mathbb{N}$). We have just observed that there is a canonical isomorphism $\iota : W \xrightarrow{\sim} G(W)$; that is, (W, ι) is a fixed point of G . The universal property is that (W, ι) is the terminal G -coalgebra. Again, this can be proved directly and follows from later theory.

(Of the many types of walk that could be considered, this one is of special interest: in a certain sense, the sequence $(W_n)_{n \geq 1}$ has period 6. See [Lei4] and compare [Bla] and [FL].)

Abstractions

In both of our examples, we characterized a topological object as the terminal coalgebra for an endofunctor. But our two examples have further features in common. We now record those features and abstract, arriving at notions of ‘discrete equational system’ and ‘universal solution’ of such a system.

In the Cantor set example, $\mathcal{C} = \mathbf{Top}$, and in the walks example, $\mathcal{C} = \mathbf{Top}^{\mathbb{N}}$. In both, then, $\mathcal{C} = \mathbf{Top}^A$ for some set A . We write objects of \mathbf{Top}^A as indexed families $(X_a)_{a \in A}$.

In the Cantor set example, the functor $G : \mathcal{C} \longrightarrow \mathcal{C}$ is defined by $G(X) = X + X$, and in the walks example, G is defined by (6). In both, G has the following property: for each $a \in A$, the space $(G(X))_a$ is a finite sum of spaces X_b ($b \in A$). More precisely, there is a family $(M_{b,a})_{b,a \in A}$ of natural numbers such that for all $X \in \mathbf{Top}^A$ and $a \in A$,

$$(G(X))_a = \sum_{b \in A} M_{b,a} \times X_b.$$

These are *finite* sums, that is, $\sum_{b \in A} M_{b,a} < \infty$ for all $a \in A$. It makes no difference for now if we take $M_{b,a}$ to be a finite set rather than a natural number, and for reasons of functoriality that emerge later, it will be better if we do so.

Thus, in both examples the category \mathcal{C} and the endofunctor G are determined by a set A and a matrix of sets $M = (M_{b,a})_{b,a \in A}$. This suggests the following definition.

Definition 1.4 A **discrete equational system** is a pair (A, M) where A is a set and M is a family $(M_{b,a})_{b,a \in A}$ of sets such that for each $a \in A$, the disjoint union $\sum_{b \in A} M_{b,a}$ is finite.

Let (A, M) be a discrete equational system and let \mathcal{E} be a category with finite sums. Then there is an endofunctor $M \otimes -$ of \mathcal{E}^A defined by

$$(M \otimes X)_a = \sum_{b \in A} M_{b,a} \times X_b \in \mathcal{E} \quad (7)$$

($X \in \mathcal{E}^A$, $a \in A$). So far we have taken $\mathcal{E} \in \mathbf{Top}$; the only other case with which we will be concerned is $\mathcal{E} = \mathbf{Set}$.

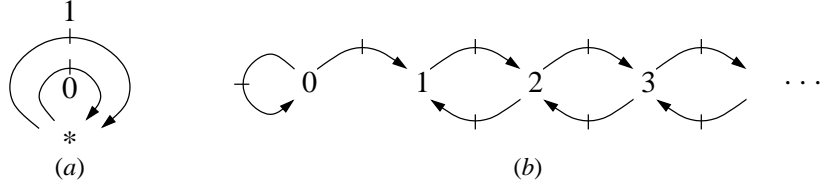


Figure 1.1: Graphs corresponding to the discrete equational systems for (a) the Cantor set (Example 1.5) and (b) the spaces of walks (Example 1.6). In (b), one walks *backwards* along the arrows

Example 1.5 (One-variable systems) A discrete equational system (A, M) in which A is a one-element set amounts to just a finite set M . If M has n elements then the induced endofunctor $M \otimes -$ of **Top** is $X \mapsto n \times X$. In the Cantor set example, $n = 2$.

Example 1.6 (Walks) The walks example corresponds to the discrete equational system (A, M) in which $A = \mathbb{N}$ and

$$|M_{b,a}| = \begin{cases} 1 & \text{if } a > 0 \text{ and } b = a \pm 1 \\ 1 & \text{if } a = b = 0 \\ 0 & \text{otherwise} \end{cases}$$

$(b, a \in \mathbb{N})$. The induced endofunctor $M \otimes -$ is exactly the functor G defined earlier.

In general, a discrete equational system can be viewed as a system of simultaneous equations using only addition, such as

$$\begin{aligned} x_0 &= x_0 \\ x_n &= x_{n-1} + x_{n+1} \end{aligned} \quad (n \in \mathbb{N}^+)$$

(the walks example), or equations (3)–(5) above. Formally, equations (3)–(5) correspond to the discrete equational system (A, M) in which $A = \{1, 2, 3\}$ and M is the transpose of the matrix of coefficients on the right-hand side: $M_{1,1} = 2$, $M_{2,1} = 5$, and so on.

A discrete equational system (A, M) can also be viewed as a graph. Call an element $m \in M_{b,a}$ a **sector** of type b in a , and write $m : b \rightarrow a$. Then there is one sector of type b in a for each copy of X_b appearing in the expression (7) for $(M \otimes X)_a$. The (directed) graph corresponding to (A, M) has the elements of A as its vertices and the sectors as its edges (Figure 1.1). The finiteness condition on M is that each $a \in A$ contains only finitely many sectors, or equivalently that each vertex is at the head of only finitely many edges.

The universal properties of the Cantor set and the spaces of walks will be expressed in the following terms.

Definition 1.7 Let (A, M) be a discrete equational system and \mathcal{E} a category with finite sums. An M -**coalgebra** (in \mathcal{E}) is a coalgebra for the endofunctor $M \otimes -$ of \mathcal{E}^A . A **universal solution** of (A, M) (in \mathcal{E}) is a terminal M -coalgebra.

Example 1.8 (Cantor set) Take the discrete equational system (A, M) of Example 1.5, with $n = 2$. The universal solution is the Cantor set $2^{\mathbb{N}^+} \in \mathbf{Top}$ together with the canonical isomorphism

$$\iota : 2^{\mathbb{N}^+} \xrightarrow{\sim} 2 \times 2^{\mathbb{N}^+} = M \otimes 2^{\mathbb{N}^+}.$$

Example 1.9 (Walks) Take the discrete equational system (A, M) of Example 1.6. The universal solution is $W = (W_n)_{n \in \mathbb{N}} \in \mathbf{Top}^{\mathbb{N}}$ together with the canonical isomorphism $\iota : W \xrightarrow{\sim} G(W) = M \otimes W$.

Universal solutions are evidently unique (up to canonical isomorphism) when they exist. The word ‘solution’ is justified by Lambek’s Lemma (1.3): if (I, ι) is a universal solution then $I \cong M \otimes I$. The converse, however, fails: for any discrete equational system (A, M) , the empty family $0 = (\emptyset)_{a \in A} \in \mathbf{Top}^A$ satisfies $0 \cong M \otimes 0$, but it is not usually the universal solution.

When \mathcal{E} is **Set** or **Top**, or more generally if \mathcal{E} has enough limits, every discrete equational system has a universal solution. This can be constructed as follows.

Let (A, M) be a discrete equational system. For each $a \in A$, let I_a be the set of all infinite sequences

$$\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0 = a$$

of sectors. Then $I \in \mathbf{Set}^A$. For each $a \in A$ we have

$$\begin{aligned} (M \otimes I)_a &= \sum_{b \in A} M_{b,a} \times I_b \\ &\cong \{ \text{diagrams } \cdots \xrightarrow{p_2} b_1 \xrightarrow{p_1} b_0 = b \xrightarrow{m} a \}, \end{aligned}$$

so there is a canonical isomorphism $\iota_a : I_a \xrightarrow{\sim} (M \otimes I)_a$. This defines an M -coalgebra (I, ι) . It can be verified directly, and follows from a more general result (Theorem 7.12), that (I, ι) is the universal solution of (A, M) in **Set**.

Moreover, each set I_a carries a natural topology, generated by declaring that for each finite diagram

$$a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0 = a, \tag{8}$$

the set of all elements of I_a ending in (8) is closed. (Denoting by $(I_n)_a$ the set of all diagrams (8), we have $I_a = \varprojlim_n (I_n)_a$, and this is the corresponding profinite topology.) Thus, (I, ι) becomes a coalgebra in **Top**. Again, it can be shown directly and follows from a later result (Theorem 8.11) that this is the universal solution in **Top**.

So, any discrete equational system specifies a family of spaces, its universal solution. But as a tool for specifying spaces, this has severe limitations: for as the construction of the universal solution (I, ι) reveals, the spaces I_a are always totally disconnected. This is a consequence of the fact that I_a is isomorphic to a *disjoint* union of the other spaces I_b :

$$I_a \cong (M \otimes I)_a = \sum_{b \in A, m: b \rightarrow a} I_b. \tag{9}$$

To specify spaces that are not totally disconnected, we will need to use non-disjoint unions, that is, glue the spaces I_b together in some nontrivial way. This is the step up from discrete equational systems to general equational systems.

2 Equational systems

A discrete equational system is a system of equations specifying each member of a family of spaces as a disjoint union of some of the others. A (general) equational system is the same, except that we are no longer confined to *disjoint* unions: any kind of union, or gluing, is permitted. Formally, this is the generalization from coproducts to colimits.

However, the process of generalization is not totally straightforward. In the general setting there are subtleties that were invisible in the discrete case, as we shall see.

The definitions are introduced by way of two examples.

The real interval

In 1999, Peter Freyd [Fre2] found a new characterization of the real interval $[0, 1]$. The interval is isomorphic to two copies of itself joined end to end, and Freyd's theorem says that it is universal as such.

The result of joining two copies of $[0, 1]$ end to end is naturally described as the interval $[0, 2]$, and then multiplication by 2 gives a bijection $[0, 1] \longrightarrow [0, 2]$, which may be written as

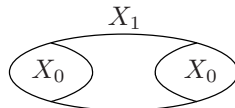
$$l_1: \bullet \text{---} \bullet \xrightarrow{\sim} \bullet \text{---} \bullet \text{---} \bullet, \quad (10)$$

The one-point space plays a role here, since that is what we are gluing along. For reasons that will become apparent, let us write

$$\iota_0 : \bullet \xrightarrow{\sim} \bullet \quad (11)$$

for the identity on the one-point space.

Now let \mathcal{C} be the category whose objects are diagrams $X_0 \begin{smallmatrix} \xrightarrow{u} \\ \xrightarrow{v} \end{smallmatrix} X_1$ where X_0 and X_1 are sets and u and v are injections with disjoint images. (For now we consider only sets; we consider spaces later.) An object $X = (X_0, X_1, u, v)$ of \mathcal{C} can be drawn as



where the copies of X_0 on the left and the right are the images of u and v respectively. A map $X \longrightarrow X'$ in \mathcal{C} consists of functions $X_0 \longrightarrow X'_0$ and $X_1 \longrightarrow X'_1$ making the evident two squares commute.

Given $X \in \mathcal{C}$, we can form a new object $G(X)$ of \mathcal{C} by gluing two copies of X end to end:

$$\begin{array}{c} X_1 \qquad X_1 \\ \text{---} \text{---} \text{---} \\ \circ \quad \circ \quad \circ \\ | \quad | \quad | \\ X_0 \quad X_0 \quad X_0 \end{array} . \tag{12}$$

Formally, the endofunctor G of \mathcal{C} is defined by pushout:

$$\begin{array}{ccccc}
 & & (G(X))_1 & & \\
 & \nearrow & & \nwarrow & \\
 & X_1 & \text{pushout} & X_1 & \\
 \nearrow u & & & & \nwarrow v \\
 (G(X))_0 = X_0 & & X_0 & & X_0.
 \end{array} \tag{13}$$

For example, the unit interval with its endpoints distinguished forms an object

$$I = \left(\{\star\} \xrightleftharpoons[1]{0} [0, 1] \right)$$

of \mathcal{C} , and

$$G(I) = \left(\{\star\} \xrightleftharpoons[2]{0} [0, 2] \right).$$

So there is a coalgebra structure $\iota : I \xrightarrow{\sim} G(I)$ on I given by (10) and (11).

Theorem 2.1 (Freyd) (I, ι) is the terminal G -coalgebra.

This follows from a later result (Example 10.5). A direct proof is not hard either, and runs roughly as follows. Take a G -coalgebra (X, ξ) and an element $x_0 \in X_1$. Then $\xi(x_0) \in (G(X))_1$ is in either the left-hand or the right-hand copy of X_1 , so gives rise to a binary digit $m_1 \in \{0, 1\}$ and a new element $x_1 \in X_1$. (If $\xi(x_0)$ is in the intersection of the two copies of X_1 , choose left or right arbitrarily.) Iterating gives a binary representation $0.m_1m_2\dots$ of an element of $[0, 1]$, and this is the image of x_0 under the unique coalgebra map $(X, \xi) \longrightarrow (I, \iota)$.

In the definition of \mathcal{C} , the condition that the maps $u, v : X_0 \longrightarrow X_1$ are injective with disjoint images is essential. Without it, the theorem would degenerate entirely: the terminal coalgebra would be $(\{\star\} \xrightarrow{\sim} \{\star\})$. As we shall see, this condition is a form of flatness. It is the source of most of the new subtleties in the non-discrete case.

There is also a topological version of Freyd's theorem. Let \mathcal{C}' be the category whose objects are diagrams $X_0 \xrightleftharpoons[v]{u} X_1$ of topological spaces and continuous closed injections with disjoint images, and whose maps are pairs of continuous maps making the evident squares commute. (A map of topological spaces is **closed** if the direct image of every closed subset is closed.) Define an endofunctor G' of \mathcal{C}' by the same pushout diagram (13) as before. Define a G' -coalgebra (I, ι) as before, with the Euclidean topology on $[0, 1]$.

Theorem 2.2 (Topological Freyd) (I, ι) is terminal in the category of G' -coalgebras.

This is proved in Example 10.5. The importance of the condition that u and v are closed is that without it, the terminal coalgebra would be given by the indiscrete topology on $[0, 1]$.

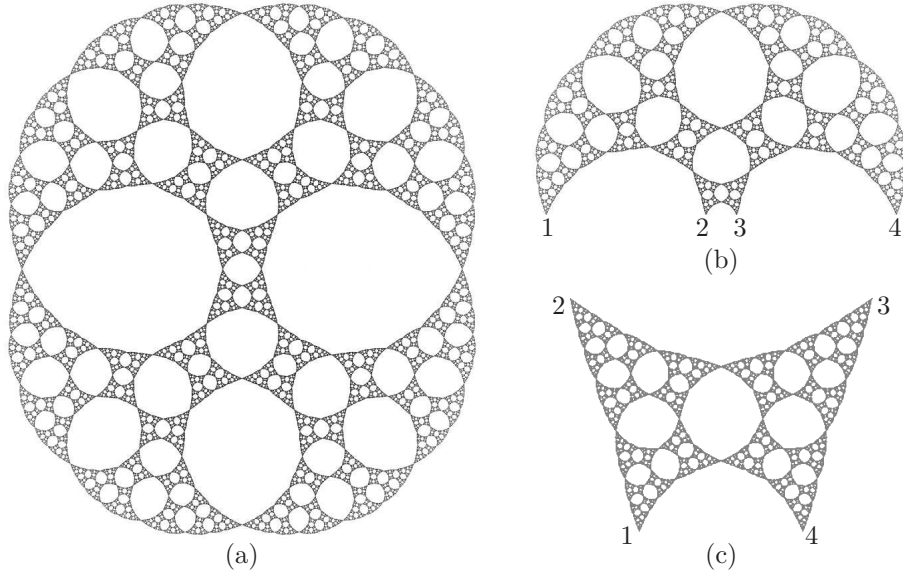


Figure 2.2: (a) The Julia set of $(2z/(1+z^2))^2$; (b), (c) two subsets, rescaled. (Image by Jon Nimmo; see also [Mil, Fig. 2] and [PR, Fig. 53])

A Julia set

The second example concerns the Julia set of a certain rational function (Figure 2.2(a)). Since the sole purpose of the example is to motivate the definitions, we will not need the definition of Julia set, and we will proceed informally. The background is that every holomorphic map $f : S \rightarrow S$ on a Riemann surface S has a Julia set $J(f) \subseteq S$; it is the part of S on which f behaves unstably under iteration. The best-explored case is where S is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and f is a rational function with complex coefficients. In this case, $J(f)$ is a closed subset of $\mathbb{C} \cup \{\infty\}$, and is almost always fractal in nature.

Figure 2.2(a) shows the Julia set of the function $z \mapsto (2z/(1+z^2))^2$. Write I_1 for this Julia set, regarded as an abstract topological space. Evidently I_1 has reflectional symmetry in a horizontal axis, so may be written

$$I_1 \cong \begin{array}{c} I_2 \\ \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \hline 1 \quad 2 \quad 3 \quad 4 \end{array} \\ I_2 \end{array} \quad (14)$$

where I_2 is a certain space with 4 distinguished points, shown in Figure 2.2(b).

In turn, I_2 may be regarded as a gluing-together of subspaces:

$$I_2 \cong \text{diagram} \quad (15)$$

where I_3 is another space with 4 distinguished points (Figure 2.2(c)). Finally, I_3 is homeomorphic to two copies of itself glued together:

$$I_3 \cong \text{diagram} \quad (16)$$

No new spaces appear at this stage, so the process ends. However, the one-point space has played a role (since we are gluing at single points), so let us write I_0 for the one-point space and record the trivial isomorphism

$$I_0 \cong I_0. \quad (17)$$

Conjecturally, the spaces I_n together with the isomorphisms (14)–(17) have the following universal property.

Let \mathcal{C} be the category whose objects are diagrams

$$\begin{array}{ccc} & X_1 & \\ & \downarrow & \\ X_0 & \begin{array}{c} \xrightarrow{u_1} \\ \xrightarrow{u_2} \\ \xrightarrow{u_3} \\ \xrightarrow{u_4} \end{array} & X_2 \\ & \searrow \begin{array}{c} \xrightarrow{v_1} \\ \xrightarrow{v_2} \\ \xrightarrow{v_3} \\ \xrightarrow{v_4} \end{array} & X_3 \end{array}$$

of topological spaces and continuous closed injections such that u_1, u_2, u_3 and u_4 have disjoint images, and similarly v_1, v_2, v_3 and v_4 . Let G be the endofunctor of \mathcal{C} corresponding to the right-hand sides of (14)–(17); for instance,

$$(G(X))_1 = \text{diagram} = (X_2 + X_2) / \sim$$

for a certain equivalence relation \sim . (The picture of $(G(X))_1$ is drawn as if X_0 were a single point.) Then, conjecturally, (14)–(17) give an isomorphism $\iota : I \xrightarrow{\sim} G(I)$ and (I, ι) is the terminal G -coalgebra. If true, this means that the simple diagrams (14)–(17) contain as much topological information as the apparently very complex spaces in Figure 2.2: given the system of equations, we recover these spaces as the universal solution. (Caveat: we consider only the intrinsic, topological aspects of the spaces, not how they are embedded into an ambient space or any metric or conformal structure.)

Abstractions

We now set out the common features of these two examples, eventually arriving at the general notion of equational system.

For both the real interval and the Julia set, the category \mathcal{C} is not \mathbf{Set}^A or \mathbf{Top}^A for any set A (as it was for discrete systems); rather, it is a full subcategory of $[\mathbb{A}, \mathbf{Set}]$ or $[\mathbb{A}, \mathbf{Top}]$ for some small category \mathbb{A} . In the case of the interval,

$$\mathbb{A} = \left(0 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} 1 \right), \quad (18)$$

and in the case of the Julia set,

$$\mathbb{A} = \left(\begin{array}{ccc} & & 1 \\ & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & 2 \\ 0 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & 3 \end{array} \right). \quad (19)$$

In neither case is \mathcal{C} the *whole* functor category $[\mathbb{A}, \mathbf{Set}]$ or $[\mathbb{A}, \mathbf{Top}]$, because of the nondegeneracy conditions on the maps u and v . A fruitful generalization of these conditions is as follows. (For the tensor product notation, see Appendix A.)

Definition 2.3 Let \mathbb{A} be a small category. A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is **nondegenerate** (or **componentwise flat**) if the functor

$$- \otimes X : [\mathbb{A}^{\text{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

preserves finite connected limits. The full subcategory of $[\mathbb{A}, \mathbf{Set}]$ formed by the nondegenerate functors is written $\langle \mathbb{A}, \mathbf{Set} \rangle$.

Write $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ for the underlying set functor. A functor $X : \mathbb{A} \longrightarrow \mathbf{Top}$ is **nondegenerate** if $U \circ X$ is nondegenerate and for each map f in \mathbb{A} , the map Xf is closed. The full subcategory of $[\mathbb{A}, \mathbf{Top}]$ formed by the nondegenerate functors is written $\langle \mathbb{A}, \mathbf{Top} \rangle$.

(In fact, it makes no difference to Definition 2.3 if we change ‘finite connected limits’ to ‘pullbacks’, by Lemma 2.1 of [CJ]. However, the class of finite connected limits is in some sense better-behaved than the class of pullbacks: see [ABLR].)

It will be shown in §4 that when \mathbb{A} is the category (18), $\langle \mathbb{A}, \mathbf{Set} \rangle$ and $\langle \mathbb{A}, \mathbf{Top} \rangle$ are the categories \mathcal{C} and \mathcal{C}' defined in the real interval example. Similar statements hold for (19) and the Julia set example.

A discrete equational system (A, M) consists of a set A and a (suitably finite) matrix M of natural numbers, that is, a map $M : A \times A \longrightarrow \mathbb{N}$. The matrix M encodes the right-hand sides of the ‘equations’ that we seek to solve, and induces an endofunctor $G = M \otimes -$ of \mathbf{Set}^A . I claim that in our two non-discrete examples, the right-hand sides are encoded by a module $M : \mathbb{A} \dashrightarrow \mathbb{A}$ (that is, a functor $M : \mathbb{A}^{\text{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$), and that the induced endofunctor $M \otimes -$ of $\langle \mathbb{A}, \mathbf{Set} \rangle$ or $\langle \mathbb{A}, \mathbf{Top} \rangle$ is the endofunctor G of our examples.

As in the discrete case, the idea is that

$$M(b, a) = \{\text{copies of the } b\text{th space used in the gluing formula for the } a\text{th space}\}$$

$(b, a \in \mathbb{A})$, and elements $m \in M(b, a)$ are called **sectors** of type b in a , written $m : b \dashrightarrow a$.

Example 2.4 (Interval) We have $\mathbb{A} = (0 \xrightleftharpoons[\tau]{\sigma} 1)$. Since, for instance, the formula (13) (or (12)) for $(G(X))_1$ contains 3 copies of X_0 , we should have $|M(0, 1)| = 3$. Naming the elements of the sets $M(b, a)$ suggestively,

$$\begin{aligned} M(0, 0) &= \{\text{id}\}, & M(0, 1) &= \{0, \tfrac{1}{2}, 1\}, \\ M(1, 0) &= \emptyset, & M(1, 1) &= \{[0, \tfrac{1}{2}], [\tfrac{1}{2}, 1]\}. \end{aligned}$$

The whole functor $M : \mathbb{A}^{\text{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$, including its action on morphisms, is defined as follows:

$$\begin{array}{ccc|ccc} & & & M(-, 0) & \xrightleftharpoons[\tau \cdot -]{\sigma \cdot -} & M(-, 1) \\ & & & \hline M(0, -) & & \{ \text{id} \} & \xrightleftharpoons[1]{0} & \{ 0, \tfrac{1}{2}, 1 \} \\ \uparrow \scriptstyle - \cdot \sigma & \uparrow & \uparrow & & \uparrow \scriptstyle \inf & \uparrow \scriptstyle \sup \\ & & M(1, -) & & \emptyset & \xrightleftharpoons{\quad} & \{ [0, \tfrac{1}{2}], [\tfrac{1}{2}, 1] \}. \end{array} \quad (20)$$

Now M is a module $\mathbb{A} \dashrightarrow \mathbb{A}$, so induces an endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$. (See Appendix A for a primer on categorical modules.) Then, for instance,

$$\begin{aligned} (M \otimes X)_1 &= (M(0, 1) \times X_0 + M(1, 1) \times X_1) / \sim \\ &= (3 \times X_0 + 2 \times X_1) / \sim \end{aligned}$$

for some equivalence relation \sim . (Compare (12) and (13).) It follows from later theory that $M \otimes -$ restricts to an endofunctor of $\mathcal{C} = \langle \mathbb{A}, \mathbf{Set} \rangle$, and this restricted endofunctor is precisely G , the endofunctor defined previously. Analogous statements hold in the topological case.

Example 2.5 (Julia set) Here \mathbb{A} is given by (19). In the gluing formula (15) for I_2 , the one-point space I_0 appears 8 times (Figure 2.3), I_1 does not appear at all, I_2 appears twice, and I_3 appears once, so

$$|M(0, 2)| = 8, \quad |M(1, 2)| = 0, \quad |M(2, 2)| = 2, \quad |M(3, 2)| = 1.$$

So, for instance, if $X \in [\mathbb{A}, \mathbf{Top}]$ then

$$(M \otimes X)_2 = (8 \times X_0 + 2 \times X_2 + X_3) / \sim$$

where \sim identifies the 8 copies of X_0 with their images in X_2 and X_3 . Again it can be shown that $M \otimes -$ restricts to an endofunctor of $\mathcal{C} = \langle \mathbb{A}, \mathbf{Top} \rangle$ and that this is the endofunctor G described earlier.

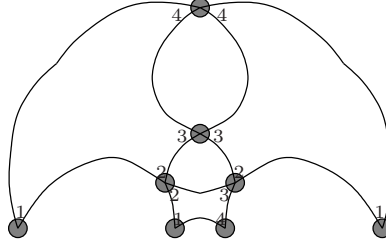


Figure 2.3: The eight elements of $M(0, 2)$

Here is an alternative way of seeing that a system of equations of this type can be expressed as a module. The right-hand sides of each of (10)–(11) and (14)–(17) are formal gluings of objects of \mathbb{A} . ‘Gluings’ are colimits, so if $\hat{\mathbb{A}}$ is the category obtained by taking \mathbb{A} and freely adjoining all possible colimits then the system of equations amounts to a functor from \mathbb{A} to $\hat{\mathbb{A}}$. But $\hat{\mathbb{A}} = [\mathbb{A}^{\text{op}}, \mathbf{Set}]$ (by [MM, I.5.4]), so the system is a functor $\mathbb{A} \longrightarrow [\mathbb{A}^{\text{op}}, \mathbf{Set}]$, that is, a module $\mathbb{A} \dashrightarrow \mathbb{A}$.

We confine ourselves to systems of equations in which the right-hand sides are *finite* gluings. To formalize this, recall that any presheaf $Y : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set}$ on a small category \mathbb{B} has a **category of elements** $\mathbb{E}(Y)$, whose objects are pairs (b, y) with $b \in \mathbb{B}$ and $y \in Y(b)$; maps $(b, y) \longrightarrow (b', y')$ are maps $g : b \longrightarrow b'$ in \mathbb{B} such that $y'g = y$. Similarly, any covariant functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ has a **category of elements** $\mathbb{E}(X)$. In each case, there is a covariant projection functor from the category of elements to \mathbb{B} or \mathbb{A} .

Definition 2.6 A presheaf $Y : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set}$ is **finite** if its category of elements is finite. A module $M : \mathbb{B} \dashrightarrow \mathbb{A}$ is **finite** if for each $a \in \mathbb{A}$, the presheaf $M(-, a)$ is finite.

Explicitly, M is finite if for each $a \in \mathbb{A}$ there are only finitely many diagrams of the form

$$b' \xrightarrow{f} b \dashrightarrow^m a.$$

Certainly this holds if, as in the interval example, the category \mathbb{A} and the sets $M(b, a)$ are finite.

Since our endofunctors $M \otimes -$ are to act on the subcategory $\langle \mathbb{A}, \mathbf{Set} \rangle$ of $[\mathbb{A}, \mathbf{Set}]$ formed by the nondegenerate functors, we need M to satisfy a further condition. Proposition 5.4 shows that the following condition is sufficient (and, in fact, necessary). Proposition 5.8 shows that also, for such an M , the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Top}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Top} \rangle$.

Definition 2.7 Let \mathbb{A} and \mathbb{B} be small categories. A module $M : \mathbb{B} \dashrightarrow \mathbb{A}$ is **nondegenerate** if $M(b, -) : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate for each $b \in \mathbb{B}$.

Definition 2.8 An **equational system** is a small category \mathbb{A} together with a finite nondegenerate module $M : \mathbb{A} \dashrightarrow \mathbb{A}$.

We might more precisely say ‘finite-colimit equational system’. The discrete equational systems are precisely the equational systems (\mathbb{A}, M) in which the category \mathbb{A} is discrete (Example 4.5).

Definition 2.9 Let (\mathbb{A}, M) be an equational system. An M -**coalgebra** in **Set** (respectively, **Top**) is a coalgebra for the endofunctor $M \otimes -$ of $\langle \mathbb{A}, \mathbf{Set} \rangle$ (respectively, $\langle \mathbb{A}, \mathbf{Top} \rangle$).

A **universal solution** of (\mathbb{A}, M) , in **Set** or **Top**, is a terminal object in the category of M -coalgebras.

Universal solutions are unique (up to isomorphism) when they exist; but just as not every ordinary system of equations has a solution, not every equational system has a universal solution. Theorem B.1 gives necessary and sufficient conditions.

For example, Freyd’s theorem (2.1) characterizes the set $[0, 1]$, together with its endpoints and the map that multiplies by two, as the universal solution in **Set** of a certain equational system. The topological Freyd theorem (2.2) characterizes the space $[0, 1]$, with the same extra structure and the Euclidean topology, as the universal solution in **Top**.

Our other example seeks to characterize a certain Julia set as (part of) the universal solution in **Top** of a certain equational system. Heuristic arguments and evidence from the theory of laminations [Thu, Kiwi] suggest a more general phenomenon. To discuss it, we need some further definitions.

Definition 2.10 A topological space S is **realizable** if there exist an equational system (\mathbb{A}, M) with universal solution (I, ι) , and an object $a \in \mathbb{A}$, such that $S \cong I(a)$. It is **discretely realizable** (respectively, **finitely realizable**) if \mathbb{A} can be taken to be discrete (respectively, finite).

(Instead of ‘realizable’, we might more precisely say ‘corecursively realizable by finite colimits’.)

Conjecture 2.11 *The Julia set $J(f)$ of any complex rational function f is finitely realizable.*

This says that in the example, we could have taken any rational function f and seen the same type of behaviour: after a finite number of decompositions, no more new spaces I_n appear. Both $J(f)$ and its complement are invariant under f , so f restricts to an endomorphism of $J(f)$, which is, with finitely many exceptions, a $\deg(f)$ -to-one mapping. This suggests that f itself should provide the recursive structure of $J(f)$, and that if (\mathbb{A}, M) is the corresponding equational system then the sizes of \mathbb{A} and M should be bounded in terms of $\deg(f)$.

Products of equational systems

We finish with some observations on products that will not be used until §10, and could be omitted on first reading.

Equational systems form a category. A **map** $(R, \rho) : (\mathbb{A}, M) \longrightarrow (\mathbb{A}', M')$ consists of a functor $R : \mathbb{A} \longrightarrow \mathbb{A}'$ together with a natural transformation

$$\begin{array}{ccc} \mathbb{A}^{\text{op}} \times \mathbb{A} & \xrightarrow{R^{\text{op}} \times R} & \mathbb{A}'^{\text{op}} \times \mathbb{A}' \\ & \searrow \scriptstyle M \quad \nearrow \scriptstyle \rho \quad \swarrow \scriptstyle M' & \\ & \mathbf{Set}. & \end{array}$$

This means that ρ assigns to each sector $b \xrightarrow{m} a$ in (\mathbb{A}, M) a sector $R(b) \xrightarrow{\rho(m)} R(a)$ in (\mathbb{A}', M') , in such a way that the equation $\rho(fmg) = R(f)\rho(m)R(g)$ is satisfied.

Lemma 2.12 (Functors on products) *Let $Z : \mathbb{B} \longrightarrow \mathbf{Set}$ and $Z' : \mathbb{B}' \longrightarrow \mathbf{Set}$ be functors on categories \mathbb{B}, \mathbb{B}' , and consider the functor*

$$\begin{array}{ccc} Z \times Z' : \mathbb{B} \times \mathbb{B}' & \longrightarrow & \mathbf{Set} \\ (b, b') & \longmapsto & Z(b) \times Z'(b'). \end{array}$$

Then:

- a. $\mathbb{E}(Z \times Z') \cong \mathbb{E}(Z) \times \mathbb{E}(Z')$
- b. *if Z and Z' are finite then so is $Z \times Z'$*
- c. *if Z and Z' are nondegenerate then so is $Z \times Z'$.*

Proof Part (a) is straightforward, and (b) follows immediately. Part (c) will follow from (a) once we have Theorem 4.11; it can also be proved by a direct calculation. \square

Now let (\mathbb{A}, M) and (\mathbb{A}', M') be equational systems. There is a module

$$M \times M' : \mathbb{A} \times \mathbb{A}' \longrightarrow \mathbb{A} \times \mathbb{A}'$$

defined by

$$(M \times M')((b, b'), (a, a')) = M(b, a) \times M'(b', a'),$$

which by Lemma 2.12 is finite and nondegenerate. So $(\mathbb{A} \times \mathbb{A}', M \times M')$ is an equational system, and it is straightforward to check that it is the product $(\mathbb{A}, M) \times (\mathbb{A}', M')$ in the category of equational systems.

Later we will use this construction to show that the product of two realizable spaces is realizable.

3 Summary of results

Now that the language of equational systems has been explained, it is possible to describe the main results of the rest of this paper. These results will give us three fundamental abilities: given an equational system (\mathbb{A}, M) , we will be able to:

- determine whether there is a universal solution
- construct the universal solution whenever it does exist
- check easily whether a given coalgebra is the universal solution.

We begin (§4) by examining the nondegeneracy condition. We give an equivalent formulation of nondegeneracy that is easy to verify in examples, unlike the original definition (2.3, 2.7).

There is a well-developed general theory of coalgebras for endofunctors, but for endofunctors $M \otimes -$ arising from equational systems, the theory has a special

flavour (§5). In a loose way it resembles homological algebra; we use terms such as *complex*, *double complex* and *resolution*. We develop this theory and prove that the endofunctor of $[\mathbb{A}, \mathbf{Set}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Set} \rangle$, and similarly for \mathbf{Top} , as was assumed in the introductory sections.

The universal solution of an equational system is quite easily described, in the case that it exists. In §6 we give explicit sufficient conditions for its existence, and construct it. In Appendix B we prove that these conditions are also necessary. Existence of a universal solution turns out to be unaffected by whether we work over \mathbf{Set} or \mathbf{Top} .

The proof that this really *is* the universal solution is substantial: §7 and §8 contain the proofs over \mathbf{Set} and \mathbf{Top} , respectively. The main tools are König's Lemma (7.1) and the homological-like algebra of coalgebras for endofunctors $M \otimes -$.

The third ability is to recognize a universal solution when we see one. We prove theorems that allow us to take a coalgebra for some equational system and decide whether it is the universal solution (§9). This is much easier than checking directly whether it matches the explicit construction.

Using these theorems we can give many examples of equational systems and their universal solutions (§10). They also let us settle the question of which topological spaces are realizable, or discretely realizable—that is, occur as one of the spaces $I(a)$ in the universal solution of some (discrete) equational system (Appendix C).

The results of this paper completely answer the most basic questions about equational systems and their universal solutions. But an important unanswered question is this: which topological spaces are *finitely* realizable? Arguably, the finite equational systems are the most interesting ones, and come closer to intuitive notions of self-similarity. But in this paper we do not attempt a serious development of the more precise theory of finite equational systems, making only the few remarks at the end of §4 and the beginning of Appendix C.

4 Nondegeneracy

The main result of this section (Theorem 4.11) is that a functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate if and only if it satisfies the following explicit conditions:

ND1 given

$$\begin{array}{ccc} a & & a' \\ & \searrow f & \swarrow f' \\ & b & \end{array}$$

in \mathbb{A} and $x \in X(a)$, $x' \in X(a')$ such that $fx = f'x'$, there exist a commutative square

$$\begin{array}{ccc} & c & \\ g \swarrow & & \searrow g' \\ a & & a' \\ f \searrow & & \swarrow f' \\ & b & \end{array}$$

and $z \in X(c)$ such that $x = gz$ and $x' = g'z$

ND2 given $a \xrightarrow[f']{f} b$ in \mathbb{A} and $x \in X(a)$ such that $fx = f'x$, there exist a fork

$$c \xrightarrow{g} a \xrightarrow[f']{f} b \quad (21)$$

and $z \in X(c)$ such that $x = gz$. (A diagram (21) is a **fork** if $fg = f'g$.)

Before developing the theory that leads up to this result, we give some examples of nondegenerate functors. They illustrate that nondegeneracy means ‘no unforced equalities’, in a sense to be explained. After the main result, we give explicit conditions for a *module* to be nondegenerate, and we look more closely at the case where \mathbb{A} is finite.

Examples of nondegenerate functors

Let us work out what nondegeneracy says for various specific categories \mathbb{A} , assuming for now that nondegeneracy is equivalent to conditions **ND1** and **ND2**.

Note that **ND1** holds automatically if either f or f' is an isomorphism, and that **ND2** holds automatically if $f = f'$. Moreover, if f is monic then **ND1** in the case $f = f'$ just says that Xf is injective.

Example 4.1 Let $\mathbb{A} = (0 \xrightarrow{\sigma} 1)$. Then $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate if and only if the function $X\sigma : X(0) \longrightarrow X(1)$ is injective.

Intuitively, nondegeneracy of a functor X says that no equation between elements of X holds unless it must. In this example, nondegeneracy of X says that the equation $\sigma x_0 = \sigma x'_0$ holds only when it must, that is, only when $x_0 = x'_0$.

Example 4.2 Let $\mathbb{A} = (0 \xrightarrow[\tau]{\sigma} 1)$, so that a functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is a pair $(X_0 \xrightarrow[X\tau]{X\sigma} X_1)$ of functions. Then **ND1** in the case $f = f'$ says that $X\sigma$ and $X\tau$ are injective. The only other nontrivial case of **ND1** is $f = \sigma$, $f' = \tau$, and since the diagram

$$\begin{array}{ccc} 0 & & 0 \\ & \searrow \sigma & \swarrow \tau \\ & 1 & \end{array}$$

cannot be completed to a commutative square, **ND1** says that $X\sigma$ and $X\tau$ have disjoint images. The only nontrivial case of **ND2** is $f = \sigma$, $f' = \tau$, and since the diagram $(0 \xrightarrow[\tau]{\sigma} 1)$ cannot be completed to a fork, this says that $\sigma x_0 \neq \tau x_0$ for all $x_0 \in X_0$, which we already know. So a nondegenerate functor on \mathbb{A} is a parallel pair of injections with disjoint images, as claimed in §2.

Example 4.3 Let \mathbb{A} be the category generated by objects and arrows

$$0 \xrightarrow[\tau]{\sigma} 1 \xrightarrow{\rho} 2$$

subject to $\rho\sigma = \rho\tau$, and consider a functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$. From **ND1** and **ND2** it follows that X is nondegenerate just when:

- $X\sigma$, $X\tau$, and $X(\rho\sigma)$ are injective
- $X\sigma$ and $X\tau$ have disjoint images
- if $\rho x_1 = \rho x'_1$ then $x_1 = x'_1$ or there exists x_0 such that $\{x_1, x'_1\} = \{\sigma x_0, \tau x_0\}$.

The last clause corresponds again to the intuition: the equation $\rho x_1 = \rho x'_1$ holds only when it must.

An example of a nondegenerate functor on \mathbb{A} is the diagram

$$\{\star\} \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} [0, 1] \longrightarrow S^1$$

exhibiting the circle as an interval with its endpoints identified.

Example 4.4 Let \mathbb{A} be the category generated by objects and arrows

$$0 \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\tau_1} \end{array} 1 \begin{array}{c} \xrightarrow{\sigma_2} \\ \xrightarrow{\tau_2} \end{array} \dots$$

subject to $\sigma_{k+1}\sigma_k = \tau_{k+1}\sigma_k$ and $\sigma_{k+1}\tau_k = \tau_{k+1}\tau_k$ for all $k \geq 1$. A functor $\mathbb{A}^{\text{op}} \longrightarrow \mathbf{Set}$ is usually called a globular set or an ω -graph. It can be shown that a coglobular set $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate precisely when:

- for all $k \geq 1$, $X\sigma_k$ and $X\tau_k$ are injective
- for all $k \geq 1$ and $x, x' \in X_k$ satisfying $\sigma_{k+1}x = \tau_{k+1}x'$, we have $x = x' \in \text{image}(X\sigma_k) \cup \text{image}(X\tau_k)$
- the images of $X\sigma_1$ and $X\tau_1$ are disjoint.

For instance, the underlying coglobular set of any disk in the sense of Joyal [Joy, Lei1] is nondegenerate.

Example 4.5 (Discrete systems) It is immediate from **ND1** and **ND2** that every **Set**- or **Top**-valued functor on a discrete category is nondegenerate. It follows that a discrete equational system is the same thing as an equational system (\mathbb{A}, M) in which the category \mathbb{A} is discrete. The categories of M -coalgebras defined in §1 and §2 then match up (working over either **Set** or **Top**); hence, so do the notions of universal solution.

Theory of nondegenerate functors

The proof of the main theorem on nondegenerate functors (4.11) uses some more sophisticated category theory than the rest of the paper. Readers who prefer to take it on trust can jump straight to the statement of the theorem.

None of this theory is new: it goes back to Grothendieck and Verdier [GV] and Gabriel and Ulmer [GU], and was later developed by Weberpals [Web], Lair [Lair], Ageron [Age], and Adámek, Borceux, Lack, and Rosický [ABLR]. More general statements of much of what follows can be found in [ABLR].

Let us begin with ordinary flat functors. A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ on a small category \mathbb{A} is **flat** if $- \otimes X : [\mathbb{A}^{\text{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ preserves finite limits. For example, representable functors are flat: if $X = \mathbb{A}(a, -)$ then $- \otimes X$ is evaluation at a , which preserves all limits.

Theorem 4.6 (Flatness) *Let \mathbb{A} be a small category. The following conditions on a functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ are equivalent:*

- a. X is flat
- b. every finite diagram in $\mathbb{E}(X)$ admits a cone
- c. each of the following holds:
 - there exists $a \in \mathbb{A}$ for which $X(a) \neq \emptyset$
 - given $a, a' \in \mathbb{A}$, $x \in X(a)$, and $x' \in X(a')$, there exist a diagram $a \xleftarrow{g} c \xrightarrow{g'} a'$ in \mathbb{A} and $z \in X(c)$ such that $gz = x$ and $g'z = x'$
 - ND2.

Proof See [Bor, §6.3] or [MM, VII.6], for instance. \square

The following lemmas are often used to prove this theorem, and will also be needed later.

Lemma 4.7 (Existence of cones) *Let \mathbb{I} and \mathbb{A} be small categories and let $X : \mathbb{A} \longrightarrow \mathbf{Set}$. Suppose that $- \otimes X : [\mathbb{A}^{\text{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ preserves limits of shape \mathbb{I} . Then every diagram of shape \mathbb{I} in $\mathbb{E}(X)$ admits a cone.*

Proof Let $D : \mathbb{I} \longrightarrow \mathbb{E}(X)$ be a diagram of shape \mathbb{I} , writing $D(i) = (a_i, x_i)$ for each $i \in \mathbb{I}$. Then there is a diagram $\mathbb{I} \longrightarrow [\mathbb{A}^{\text{op}}, \mathbf{Set}]$ given by $i \mapsto \mathbb{A}(-, a_i)$, so by hypothesis the canonical map

$$\left(\lim_{\leftarrow i} \mathbb{A}(-, a_i) \right) \otimes X \longrightarrow \lim_{\leftarrow i} (\mathbb{A}(-, a_i) \otimes X) \cong \lim_{\leftarrow i} X(a_i)$$

is a bijection, and in particular a surjection. With the usual explicit formula for limits in \mathbf{Set} , this map is

$$(a \xrightarrow{p_i} a_i)_{i \in \mathbb{I}} \otimes x \longmapsto (p_i x)_{i \in \mathbb{I}}$$

where $a \in \mathbb{A}$, $x \in X(a)$ and

$$(a \xrightarrow{p_i} a_i)_{i \in \mathbb{I}} \in \{\text{cones from } a \text{ to } (a_i)_{i \in \mathbb{I}}\} = \lim_{\leftarrow i} \mathbb{A}(a, a_i).$$

Since $(x_i)_{i \in \mathbb{I}} \in \lim_{\leftarrow i} X(a_i)$, there exist $a \in \mathbb{A}$ and

$$((p_i)_{i \in \mathbb{I}}, x) \in \left(\lim_{\leftarrow i} \mathbb{A}(a, a_i) \right) \times X(a)$$

such that $p_i x = x_i$ for all i . Then $\left((a, x) \xrightarrow{p_i} (a_i, x_i) \right)_{i \in \mathbb{I}}$ is a cone on D . \square

Let us say that a category \mathcal{C} has the **square-completion property** if there exists a cone on every diagram of shape $(\bullet \longrightarrow \bullet \longleftarrow \bullet)$ in \mathcal{C} .

Lemma 4.8 (Connectedness by spans) *Two objects c, c' of a category with the square-completion property are in the same connected-component if and only if there exists a span $c \longleftarrow c'' \longrightarrow c'$ connecting them.* \square

Lemma 4.9 (Equality in a tensor product) *Let \mathbb{A} be a small category and*

$$X : \mathbb{A} \longrightarrow \mathbf{Set}, \quad Y : \mathbb{A}^{\text{op}} \longrightarrow \mathbf{Set}.$$

Suppose that $\mathbb{E}(X)$ has the square-completion property. Let

$$a, a' \in \mathbb{A}, \quad (y, x) \in Y(a) \times X(a), \quad (y', x') \in Y(a') \times X(a').$$

Then $y \otimes x = y' \otimes x' \in Y \otimes X$ if and only if there exist a span $a \xleftarrow{f} b \xrightarrow{f'} a'$ and an element $z \in X(b)$ such that $x = fz$, $x' = f'z$, and $yf = y'f'$.

Proof See the remarks after the statement of Theorem VII.6.3 in [MM]. \square

We need a fact about connectedness.

Lemma 4.10 (Components of a functor) *Any functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ on a small category \mathbb{A} can be written as a sum $X \cong \sum_{j \in J} X_j$, where J is some set and $\mathbb{E}(X_j)$ is connected for each $j \in J$.*

Proof We use the equivalence between **Set**-valued functors and discrete opfibrations. Write $\mathbb{E}(X)$ as a sum $\sum_{j \in J} \mathbb{E}_j$ of connected categories. For each j , the restriction to \mathbb{E}_j of the projection $\mathbb{E}(X) \longrightarrow \mathbb{A}$ is still a discrete opfibration, so corresponds to a functor $X_j : \mathbb{A} \longrightarrow \mathbf{Set}$. Then

$$\mathbb{E}\left(\sum X_j\right) \cong \sum \mathbb{E}(X_j) \cong \sum \mathbb{E}_j \cong \mathbb{E}(X)$$

compatibly with the projections, so $\sum X_j \cong X$. \square

Here is the main result.

Theorem 4.11 (Nondegenerate functors) *Let \mathbb{A} be a small category. The following conditions on a functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ are equivalent:*

- a. *X is nondegenerate*
- b. *every finite connected diagram in $\mathbb{E}(X)$ admits a cone*
- c. *X satisfies **ND1** and **ND2***
- d. *X is a sum of flat functors.*

Remark In Lemma 4.10, the functors X_j may be regarded as the connected-components of X . A further equivalent condition is that every connected-component of X is flat: hence the name ‘componentwise flat’.

Proof

(a) \implies (b) Follows from Lemma 4.7.

(b) \implies (c) **ND1** says that every diagram of shape $(\bullet \longrightarrow \bullet \longleftarrow \bullet)$ in $\mathbb{E}(X)$ admits a cone, and similarly **ND2** for $(\bullet \rightrightarrows \bullet)$.

(c) \implies (d) Write $X \cong \sum_{j \in J} X_j$ as in Lemma 4.10. Then in each $\mathbb{E}(X_j)$, there exists a cone on every diagram of shape

$$(\bullet \longrightarrow \bullet \longleftarrow \bullet) \quad \text{or} \quad (\bullet \rightrightarrows \bullet)$$

(since $\mathbb{E}(X) \cong \sum_j \mathbb{E}(X_j)$), of shape \emptyset (since $\mathbb{E}(X_j)$ is connected and therefore nonempty), and of shape $(\bullet \quad \bullet)$ (since $\mathbb{E}(X_j)$ is connected and has the square-completion property). So by (c) \implies (a) of Theorem 4.6, each X_j is flat.

(d) \implies (a) Sums commute with connected limits in **Set**, so any sum of nondegenerate functors is nondegenerate. \square

Corollary 4.12 (Componentwise filtered categories) *The following conditions on a small category \mathbb{B} are equivalent:*

- a. *finite connected limits commute with colimits of shape \mathbb{B} in **Set***
- b. *every finite connected diagram in \mathbb{B} admits a cocone*
- c. *every diagram $b_1 \longleftarrow b_3 \longrightarrow b_2$ in \mathbb{B} can be completed to a commutative square, and every parallel pair $b_1 \xrightarrow{f} b_2 \xleftarrow{f'}$ of arrows in \mathbb{B} can be extended to a cofork.*

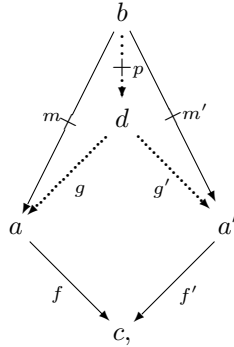
Proof In Theorem 4.11, take $\mathbb{A} = \mathbb{B}^{\text{op}}$ and X to be the functor with constant value 1. Then $\mathbb{E}(X) \cong \mathbb{B}^{\text{op}}$ and $- \otimes X \cong \lim_{\rightarrow \mathbb{B}}$. The result follows. \square

A small category \mathbb{B} satisfying the equivalent conditions of Corollary 4.12 is called **componentwise filtered**, since a further equivalent condition is that every connected-component is filtered. (Grothendieck and Verdier call such categories ‘pseudo-filtrantes’ [GV].) So $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate just when $\mathbb{E}(X)$ is componentwise cofiltered.

Nondegenerate modules

We now give a diagrammatic formulation of nondegeneracy of a *module*. This will be invaluable later. By Theorem 4.11, a module $M : \mathbb{A} \dashrightarrow \mathbb{A}$ is nondegenerate if and only if:

ND1 any commutative square of solid arrows



can be filled in by dotted arrows to a commutative diagram as shown, and

ND2 any diagram $b \xrightarrow{m} a \xrightleftharpoons[f']{f} c$ with $fm = f'm$ can be extended to a diagram

$$\begin{array}{ccc}
 & & d \\
 & \nearrow p & \vdots e \\
 b & \xrightarrow{m} & a \\
 & \searrow f & \downarrow f' \\
 & & c
 \end{array}$$

in which the triangle commutes and the right-hand column is a fork.

Finite equational systems

When the category \mathbb{A} is finite, as is often the case in examples of equational systems (\mathbb{A}, M) , some more precise results can be proved. They will not be used in the main development of the theory, but nevertheless shed light on the concept of nondegeneracy. I thank André Joyal for bringing them to my attention.

We will use the categorical notion of Cauchy-completeness. An **idempotent** in a category \mathbb{B} is an endomorphism $e : b \longrightarrow b$ in \mathbb{B} such that $e^2 = e$. It **splits** if there exist maps $a \xrightleftharpoons[p]{i} b$ such that $pi = 1_a$ and $ip = e$. A category \mathbb{B} is **Cauchy-complete** (or **Karoubi closed**) if every idempotent in \mathbb{B} splits. The importance of this condition is explained in [Law] and [Bor]. Every example of a category \mathbb{A} in this paper is Cauchy-complete.

Lemma 4.13 *Let \mathbb{B} be a Cauchy-complete category and $X : \mathbb{B} \longrightarrow \mathbf{Set}$ a finite functor (that is, a functor whose category of elements is finite). Then X is flat $\iff X$ is representable.*

Proof See Lemma 5.2 of [Lei5], for instance. \square

Lemma 4.14 *Let \mathbb{B} be a finite category and $X : \mathbb{B} \longrightarrow \mathbf{Set}$ a flat functor. Then X is finite.*

Proof Write N for the number of arrows in \mathbb{B} : then every object in $\mathbb{E}(X)$ is the domain of at most N arrows. Let S be any finite set of objects of $\mathbb{E}(X)$. Since X is flat, $\mathbb{E}(X)$ is cofiltered, so there is a cone on S in $\mathbb{E}(X)$. Its vertex is the domain of at least $|S|$ arrows, so $|S| \leq N$. Hence $\mathbb{E}(X)$ has at most N objects. Finally, the hom-sets of $\mathbb{E}(X)$ are finite, since the same is true in \mathbb{B} ; so $\mathbb{E}(X)$ is finite. \square

Proposition 4.15 (Flat functors on finite categories) *Let \mathbb{B} be a finite Cauchy-complete category and $X : \mathbb{B} \longrightarrow \mathbf{Set}$ a functor. Then:*

- a. X is flat $\iff X$ is representable
- b. X is nondegenerate $\iff X$ is a sum of representables.

Proof For (a), combine Lemmas 4.13 and 4.14. Part (b) follows, using Theorem 4.11. \square

Example 4.16 Consider the Freyd equational system (\mathbb{A}, M) (§2). The category \mathbb{A} is finite and Cauchy-complete, so Proposition 4.15 applies. Hence the functors

$$M(0, -), M(1, -), I : \mathbb{A} \longrightarrow \mathbf{Set}$$

are all sums of representables. Indeed,

$$\begin{aligned} M(0, -) &= (\{\text{id}\} \rightrightarrows \{0, \tfrac{1}{2}, 1\}) && \cong \mathbb{A}(0, -) + \mathbb{A}(1, -), \\ M(1, -) &= (\emptyset \rightrightarrows \{[0, \tfrac{1}{2}], [\tfrac{1}{2}, 1]\}) && \cong 2\mathbb{A}(1, -), \\ I &= (\{\star\} \rightrightarrows [0, 1]) && \cong \mathbb{A}(0, -) + (0, 1) \times \mathbb{A}(1, -) \end{aligned}$$

where $(0, 1)$ is the open real interval.

5 Coalgebras

The general theory of coalgebras for endofunctors has been studied extensively: see [Adá], for instance. But it turns out that coalgebras can be understood particularly well when the endofunctor is presented as

$$M \otimes - : [\mathbb{A}, \mathbf{Set}] \longrightarrow [\mathbb{A}, \mathbf{Set}]$$

for some small category \mathbb{A} and module $M : \mathbb{A} \dashrightarrow \mathbb{A}$. (Every colimit-preserving endofunctor of $[\mathbb{A}, \mathbf{Set}]$ has a unique such presentation.) We begin this section with some results about coalgebras for such endofunctors. These results will be used later, and have also been used in the pure theory of coalgebras [KMV].

We then restrict to the situation where (\mathbb{A}, M) is an equational system, and discharge our obligation to prove that $M \otimes -$ defines an endofunctor on the categories $\langle \mathbb{A}, \mathbf{Set} \rangle$ and $\langle \mathbb{A}, \mathbf{Top} \rangle$ of nondegenerate functors.

The key concept throughout is *resolution*.

Resolutions

A coalgebra can be thought of as a kind of iterative system [Adá]. To see this in our context, let \mathbb{A} be any small category, $M : \mathbb{A} \dashrightarrow \mathbb{A}$ any module, and (X, ξ) a coalgebra for the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$. Let $a_0 \in \mathbb{A}$ and $x_0 \in X(a_0)$. The map

$$\xi_{a_0} : X(a_0) \longrightarrow (M \otimes X)(a_0) = \left(\sum_{a_1} M(a_1, a_0) \times X(a_1) \right) / \sim$$

sends x_0 to

$$\xi_{a_0}(x_0) = (a_1 \dashrightarrow^{m_1} a_0) \otimes x_1$$

for some $a_1 \in \mathbb{A}$, $m_1 \in M(a_1, a_0)$ and $x_1 \in X(a_1)$. (To represent $\xi_{a_0}(x_0)$ as $m_1 \otimes x_1$ requires a choice; there are in general many such representations.) Similarly, we may write

$$\xi_{a_1}(x_1) = (a_2 \dashrightarrow^{m_2} a_1) \otimes x_2.$$

Continuing in this way, we obtain a diagram

$$\cdots \dashrightarrow^{m_{n+1}} a_n \dashrightarrow^{m_n} \cdots \dashrightarrow^{m_2} a_1 \dashrightarrow^{m_1} a_0 \tag{22}$$

and a sequence $x_\bullet = (x_n)_{n \in \mathbb{N}}$ with $x_n \in X(a_n)$ and

$$\xi_{a_n}(x_n) = m_{n+1} \otimes x_{n+1}$$

for all $n \in \mathbb{N}$. The diagram (22) together with the sequence x_\bullet will be called a **resolution** $(a_\bullet, m_\bullet, x_\bullet)$ of x_0 . I will also call x_\bullet a resolution of x_0 **along** the diagram (22).

Clearly every element x of a coalgebra has at least one resolution. But to what extent are resolutions unique? We cannot expect there to be, literally, a unique resolution of x , since at each step there is some choice in how to represent $\xi_{a_n}(x_n)$. However, we might hope that the various resolutions of x are related in some way. This is indeed the case, as we shall see, when the functor X is nondegenerate.

We begin by describing how much choice is involved in each individual step.

Lemma 5.1 (Equality in $M \otimes X$) *Let \mathbb{A} be a small category, let $M : \mathbb{A} \rightarrow \mathbb{A}$, and let $X \in \langle \mathbb{A}, \mathbf{Set} \rangle$. Take module elements*

$$\begin{array}{ccc} b & & b' \\ & \searrow m & \swarrow m' \\ & a & \end{array}$$

and $x \in X(b)$, $x' \in X(b')$. Then $m \otimes x = m' \otimes x' \in (M \otimes X)(a)$ if and only if there exist a commutative square

$$\begin{array}{ccc} & c & \\ f \swarrow & & \searrow f' \\ b & & b' \\ & \searrow m & \swarrow m' \\ & a & \end{array}$$

and an element $z \in X(c)$ such that $fz = x$ and $f'z = x'$.

Proof By Theorem 4.11, $\mathbb{E}(X)$ has the square-completion property. Now apply Lemma 4.9 with $Y = M(-, a)$. \square

We will need some terminology. Let \mathbb{A} be a small category and $M : \mathbb{A} \rightarrow \mathbb{A}$ a module.

A **complex** in (\mathbb{A}, M) is a diagram (22), abbreviated as (a_\bullet, m_\bullet) . A **map** $(a_\bullet, m_\bullet) \rightarrow (a'_\bullet, m'_\bullet)$ of complexes is a sequence $f_\bullet = (f_n)_{n \in \mathbb{N}}$ of maps in \mathbb{A} such that the diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{m_{n+1}} & a_n & \xrightarrow{m_n} & \cdots & \xrightarrow{m_2} & a_1 \xrightarrow{m_1} a_0 \\ & & \downarrow f_n & & & \downarrow f_1 & \downarrow f_0 \\ \cdots & \xrightarrow{m'_{n+1}} & a'_n & \xrightarrow{m'_n} & \cdots & \xrightarrow{m'_2} & a'_1 \xrightarrow{m'_1} a'_0 \end{array}$$

commutes. For each $a \in \mathbb{A}$ there is a category $\mathcal{J}(a)$ whose objects are the complexes (a_\bullet, m_\bullet) satisfying $a_0 = a$, and whose maps f_\bullet are those satisfying $f_0 = 1_a$.

Now let (X, ξ) be a coalgebra, $a \in \mathbb{A}$, and $x \in X(a)$. There is a category $\mathbf{Reso}(x)$ whose objects are resolutions $(a_\bullet, m_\bullet, x_\bullet)$ of x , and whose maps

$$(a_\bullet, m_\bullet, x_\bullet) \longrightarrow (a'_\bullet, m'_\bullet, x'_\bullet)$$

are the maps $f_\bullet : (a_\bullet, m_\bullet) \longrightarrow (a'_\bullet, m'_\bullet)$ in $\mathcal{I}(a)$ such that $f_n x_n = x'_n$ for all $n \in \mathbb{N}$.

Proposition 5.2 (Essential uniqueness of resolutions) *Let \mathbb{A} be a small category, $M : \mathbb{A} \dashrightarrow \mathbb{A}$ a module, and (X, ξ) a coalgebra for the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$, with X nondegenerate. Let $a \in \mathbb{A}$ and $x \in X(a)$. Then the category $\mathbf{Reso}(x)$ is connected.*

Remark In fact, $\mathbf{Reso}(x)$ is cofiltered, as can be proved by an easy extension of the argument below. We will not need this sharper result.

Proof Certainly $\mathbf{Reso}(x)$ is nonempty. Now take resolutions $(a_\bullet, m_\bullet, x_\bullet)$ and $(a'_\bullet, m'_\bullet, x'_\bullet)$ of x . We will construct a span

$$(a_\bullet, m_\bullet, x_\bullet) \longleftarrow (b_\bullet, p_\bullet, y_\bullet) \longrightarrow (a'_\bullet, m'_\bullet, x'_\bullet)$$

in $\mathbf{Reso}(x)$. Such a span consists of a commutative diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{m_3} & a_2 & \xrightarrow{m_2} & a_1 & \xrightarrow{m_1} & a_0 = a \\ & & \uparrow f_2 & & \uparrow f_1 & & \uparrow f_0 = 1_a \\ \cdots & \xrightarrow{p_3} & b_2 & \xrightarrow{p_2} & b_1 & \xrightarrow{p_1} & b_0 = a \\ & & \downarrow f'_2 & & \downarrow f'_1 & & \downarrow f'_0 = 1_a \\ \cdots & \xrightarrow{m'_3} & a'_2 & \xrightarrow{m'_2} & a'_1 & \xrightarrow{m'_1} & a'_0 = a \end{array}$$

and a sequence $(y_n)_{n \in \mathbb{N}}$ with $y_n \in X(b_n)$, such that $y_0 = x$ and

$$\xi_{b_n}(y_n) = p_{n+1} \otimes y_{n+1}, \quad f_n y_n = x_n, \quad f'_n y_n = x'_n$$

for each $n \in \mathbb{N}$.

Suppose inductively that $n \in \mathbb{N}$ and b_r, p_r, y_r, f_r and f'_r have been constructed for all $r \leq n$. We may write

$$\xi(y_n) = (c \xrightarrow{q} b_n) \otimes z$$

for some $c \in \mathbb{A}$ and $z \in X(c)$. Then

$$\xi(x_n) = \xi(f_n y_n) = f_n \xi(y_n) = f_n (q \otimes z) = (f_n q) \otimes z,$$

but also $\xi(x_n) = m_{n+1} \otimes x_{n+1}$, so by nondegeneracy of X and Lemma 5.1, there exist a commutative diagram as labelled (a) below and an element $w \in X(d)$

such that $gw = x_{n+1}$ and $hw = z$:

$$\begin{array}{ccccc}
 & & a_{n+1} & \xrightarrow{m_{n+1}} & a_n \\
 & & \uparrow g & & \uparrow f_n \\
 & & d & \xrightarrow{h} & c \\
 & \nearrow k & & & \xrightarrow{q} \\
 b_{n+1} & & & & b_n \\
 & \searrow k' & & & \downarrow f'_n \\
 & & d' & \xrightarrow{h'} & c \\
 & & \downarrow g' & & \downarrow f'_n \\
 & & a'_{n+1} & \xrightarrow{m'_{n+1}} & a'_n
 \end{array}
 \quad \begin{array}{l} \text{(a)} \\ \text{(b)} \\ \text{(a')} \end{array}$$

Similarly, there exist a commutative diagram (a') and $w' \in X(d')$ such that $g'w' = x'_{n+1}$ and $h'w' = z$. So by nondegeneracy of X (condition **ND1**), there exist a commutative square (b) and $y_{n+1} \in X(b_{n+1})$ such that $ky_{n+1} = w$ and $k'y_{n+1} = w'$. Put $p_{n+1} = qhk$, $f_{n+1} = gk$, and $f'_{n+1} = g'k'$: then

$$\xi_{b_n}(y_n) = q \otimes z = q \otimes hw = q \otimes hky_{n+1} = qhk \otimes y_{n+1} = p_{n+1} \otimes y_n,$$

and the inductive construction is complete. \square

Corollary 5.3 (Resolving complex) *Take (\mathbb{A}, M) , (X, ξ) , $a \in \mathbb{A}$ and $x \in X(a)$ as in Proposition 5.2. Then any two complexes along which x can be resolved lie in the same connected-component of $\mathcal{J}(a)$.*

Proof The complexes along which x can be resolved are the objects of $\mathcal{J}(a)$ in the image of the forgetful functor $\mathbf{Reso}(x) \longrightarrow \mathcal{J}(a)$. \square

Hence, assuming that the functor X is nondegenerate (and with no assumptions on \mathbb{A} and M), each element $x \in X(a)$ gives rise canonically to a connected-component of complexes ending at a .

From the perspective of computer science, a complex along which x can be resolved may be thought of as the observed behaviour of x under iterated application of ξ . The corollary states that any two observed behaviours are equivalent.

Coalgebras for nondegenerate modules

We still have to prove that for any equational system (\mathbb{A}, M) , the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Set} \rangle$, and similarly with **Top** in place of **Set**. The set-theoretic case is straightforward.

Proposition 5.4 (Set-theoretic endofunctor) *Let \mathbb{A} be a small category and $M : \mathbb{A} \dashrightarrow \mathbb{A}$ a nondegenerate module. Then the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Set} \rangle$.*

Nondegeneracy of M is also a *necessary* condition, since for each $b \in \mathbb{A}$ the representable $\mathbb{A}(b, -)$ is nondegenerate, and $M \otimes \mathbb{A}(b, -) = M(b, -)$.

Proof Let $X : \mathbb{A} \longrightarrow \mathbf{Set}$ be nondegenerate. Then for any finite connected limit $\lim_{\leftarrow i} Y_i$ in $[\mathbb{A}^{\text{op}}, \mathbf{Set}]$,

$$\lim_{\leftarrow i} (Y_i \otimes M \otimes X) \cong \left(\lim_{\leftarrow i} (Y_i \otimes M) \right) \otimes X \cong \left(\lim_{\leftarrow i} Y_i \right) \otimes M \otimes X,$$

the first isomorphism by nondegeneracy of X and the second by nondegeneracy of M . So $M \otimes X$ is nondegenerate. \square

We now begin the topological case.

Lemma 5.5 (Closed quotient map) *Let \mathbb{A} be a small category, $X : \mathbb{A} \longrightarrow \mathbf{Top}$ a nondegenerate functor, and $Y : \mathbb{A}^{\text{op}} \longrightarrow \mathbf{Set}$ a finite functor. Then the quotient map*

$$q : \sum_a Y(a) \times X(a) \longrightarrow Y \otimes X$$

is closed.

Proof A subset of $Y \otimes X$ is closed just when its inverse image under q is closed, so we must show that if V is a closed subset of $\sum Y(a) \times X(a)$ then its saturation $q^{-1}qV$ is also closed. Given $a \in \mathbb{A}$ and $y \in Y(a)$, write $V_{a,y} \subseteq X(a)$ for the intersection of V with the (a, y) -summand $X(a)$ of

$$\sum_{(a,y) \in \mathbb{E}(Y)} X(a) \cong \sum_{a \in \mathbb{A}} Y(a) \times X(a).$$

Then $q^{-1}qV = \bigcup_{(a,y) \in \mathbb{E}(Y)} q^{-1}qV_{a,y}$, so by finiteness of Y it suffices to show that each set $q^{-1}qV_{a,y}$ is closed.

Fix $(a, y) \in \mathbb{E}(Y)$. By definition,

$$q^{-1}qV_{a,y} = \{(a', y', x') \in \sum_{a' \in \mathbb{A}} Y(a') \times X(a') \mid y' \otimes x' = y \otimes x \text{ for some } x \in V_{a,y}\}.$$

So by nondegeneracy of X and Lemma 5.1, $(a', y', x') \in q^{-1}qV_{a,y}$ if and only if:

there exist a span

$$\begin{array}{ccc} & b & \\ f \swarrow & & \searrow f' \\ a & & a' \end{array}$$

in \mathbb{A} and $z \in X(b)$ such that $fz \in V_{a,y}$, $f'z = x'$, and $yf = y'f'$

or equivalently:

there exist a span

$$\begin{array}{ccc} & (b, w) & \\ f \swarrow & & \searrow f' \\ (a, y) & & (a', y') \end{array} \tag{23}$$

in $\mathbb{E}(Y)$ and $z \in X(b)$ such that $fz \in V_{a,y}$ and $f'z = x'$.

So

$$q^{-1}qV_{a,y} = \bigcup_{\text{spans (23)}} \{(a', y', x') \mid x' \in (Xf')(Xf)^{-1}V_{a,y}\}.$$

But $V_{a,y}$ is closed in $X(a)$, each Xf is continuous, and each Xf' is closed, so each of the sets $\{\dots\}$ in this union is a closed subset of the (a', y') -summand $X(a')$. Moreover, finiteness of Y guarantees that the union is finite. Hence $q^{-1}qV_{a,y}$ is closed, as required. \square

Part of the definition of nondegeneracy of a functor $X : \mathbb{A} \longrightarrow \mathbf{Top}$ is that for each map f in \mathbb{A} , the map Xf is closed. In later theory we will never use this condition directly; we will only use the property described in the lemma.

Corollary 5.6 (Coprojections closed) *Let \mathbb{A} be a small category, $M : \mathbb{A} \dashrightarrow \mathbb{A}$ a finite module, and $X : \mathbb{A} \longrightarrow \mathbf{Top}$ a nondegenerate functor. Then for each $m : b \dashrightarrow a$ in M , the coprojection*

$$m \otimes - : X(b) \longrightarrow (M \otimes X)(a)$$

is closed.

Proof The map $m \otimes -$ is the composite

$$X(b) \xrightarrow{(m, -)} \sum_{b'} M(b', a) \times X(b') \xrightarrow{\text{quotient map}} (M \otimes X)(a).$$

The first map is closed since it is a coproduct-coprojection, and the second is closed by Lemma 5.5. \square

Lemma 5.7 (Change of category) *Let \mathbb{A} be a small category and $M : \mathbb{A} \dashrightarrow \mathbb{A}$ a finite module. Let \mathcal{E} and \mathcal{E}' be categories with finite colimits, and $F : \mathcal{E} \longrightarrow \mathcal{E}'$ a functor preserving finite colimits. Then the square*

$$\begin{array}{ccc} [\mathbb{A}, \mathcal{E}] & \xrightarrow{M \otimes -} & [\mathbb{A}, \mathcal{E}] \\ F \circ - \downarrow & & \downarrow F \circ - \\ [\mathbb{A}, \mathcal{E}'] & \xrightarrow{M \otimes -} & [\mathbb{A}, \mathcal{E}'], \end{array}$$

commutes up to canonical isomorphism.

Proof Straightforward. \square

Proposition 5.8 (Topological endofunctor) *Let (\mathbb{A}, M) be an equational system. Then the endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Top}]$ restricts to an endofunctor of $\langle \mathbb{A}, \mathbf{Top} \rangle$.*

Proof Let $X \in \langle \mathbb{A}, \mathbf{Top} \rangle$. The functor $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ preserves colimits (being left adjoint to the indiscrete space functor), so $M \otimes (U \circ X) \cong U \circ (M \otimes X)$ by Lemma 5.7. But $M \otimes (U \circ X)$ is nondegenerate by Proposition 5.4, so $U \circ (M \otimes X)$ is nondegenerate.

Now let $a \xrightarrow{f} a'$ be a map in \mathbb{A} , and consider the commutative square

$$\begin{array}{ccc} \sum_b M(b, a) \times X(b) & \xrightarrow{\sum f_* \times 1} & \sum_b M(b, a') \times X(b) \\ q_a \downarrow & & \downarrow q_{a'} \\ (M \otimes X)(a) & \xrightarrow{(M \otimes X)f} & (M \otimes X)(a'). \end{array}$$

The map $\sum f_* \times 1$ is closed because each set $M(b, a)$ is finite. The map $q_{a'}$ is closed by Lemma 5.5 and finiteness of M . So $((M \otimes X)f) \circ q_a$ is closed; but q_a is a continuous surjection, so $(M \otimes X)f$ is closed. \square

We have now shown that for an equational system (\mathbb{A}, M) , there are induced endofunctors $M \otimes -$ of both $\langle \mathbb{A}, \mathbf{Set} \rangle$ and $\langle \mathbb{A}, \mathbf{Top} \rangle$. We will study the categories of coalgebras of these endofunctors, denoted $\mathbf{Coalg}(M, \mathbf{Set})$ and $\mathbf{Coalg}(M, \mathbf{Top})$.

The forgetful functor $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ induces a functor

$$U_* : \mathbf{Coalg}(M, \mathbf{Top}) \longrightarrow \mathbf{Coalg}(M, \mathbf{Set}).$$

Indeed, if (X, ξ) is an M -coalgebra in \mathbf{Top} then $U \circ X : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate, and by Lemma 5.7 there is a natural transformation

$$U\xi : U \circ X \longrightarrow U \circ (M \otimes X) \cong M \otimes (U \circ X).$$

Proposition 5.9 (Top vs Set) *Let (\mathbb{A}, M) be an equational system. The forgetful functor*

$$U_* : \mathbf{Coalg}(M, \mathbf{Top}) \longrightarrow \mathbf{Coalg}(M, \mathbf{Set})$$

has a left adjoint, and if (I, ι) is a universal solution in \mathbf{Top} then $U_(I, \iota)$ is a universal solution in \mathbf{Set} .*

Conversely, we will see later that any universal solution in \mathbf{Set} carries a natural topology, and is then the universal solution in \mathbf{Top} .

Proof Let D be the left adjoint to $U : \mathbf{Top} \longrightarrow \mathbf{Set}$, assigning to each set the corresponding discrete space. Then composition with D induces a functor $\langle \mathbb{A}, \mathbf{Set} \rangle \longrightarrow \langle \mathbb{A}, \mathbf{Top} \rangle$. Moreover, D preserves colimits, so commutes with $M \otimes -$ (Lemma 5.7); hence D also induces a functor

$$D_* : \mathbf{Coalg}(M, \mathbf{Set}) \longrightarrow \mathbf{Coalg}(M, \mathbf{Top}).$$

For purely formal reasons, the adjunction $D \dashv U$ induces an adjunction $D_* \dashv U_*$. The statement on universal solutions follows from the fact that right adjoints preserve terminal objects. \square

Example 5.10 (Discrete systems) When \mathbb{A} is discrete, most of the results of this section become trivial. Every \mathbf{Set} - or \mathbf{Top} -valued functor on a discrete category is nondegenerate, so $\langle \mathbb{A}, \mathbf{Set} \rangle = [\mathbb{A}, \mathbf{Set}]$ and $\langle \mathbb{A}, \mathbf{Top} \rangle = [\mathbb{A}, \mathbf{Top}]$.

Let $M : \mathbb{A} \dashrightarrow \mathbb{A}$ be a module and (X, ξ) an M -coalgebra in \mathbf{Set} . Then every element $x \in X(a)$ ($a \in \mathbb{A}$) has a unique resolution, and $\mathbf{Reso}(x)$ is the terminal category $\mathbf{1}$. As we saw in §1, every discrete equational system (\mathbb{A}, M) has a universal solution in both \mathbf{Top} and \mathbf{Set} ; and in accordance with Proposition 5.9, the universal solution in \mathbf{Top} is the universal solution in \mathbf{Set} , suitably topologized.

6 Construction of the universal solution

In this section we construct the universal solutions in **Set** and in **Top** of any given equational system, assuming that the system satisfies a certain solvability condition **S**. In Appendix B, this sufficient condition is shown to be necessary: **S** holds if and only if there is a universal solution in **Set**, if and only if there is a universal solution in **Top**. The construction therefore gives the universal solution whenever one exists. This is very unusual in the theory of coalgebras: in many contexts, sufficient conditions are known for the existence of a terminal coalgebra, but few are known to be necessary. Compare also [KMV].

Condition **S** on an equational system (\mathbb{A}, M) is:

S1 for every commutative diagram

$$\begin{array}{ccccccc}
 & & m_3 & & m_2 & & m_1 \\
 \cdots & \xrightarrow{\quad} & a_2 & \xrightarrow{\quad} & a_1 & \xrightarrow{\quad} & a_0 \\
 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\
 \cdots & \xrightarrow{\quad} & b_2 & \xrightarrow{\quad} & b_1 & \xrightarrow{\quad} & b_0 \\
 & & \uparrow f'_2 & & \uparrow f'_1 & & \uparrow f'_0 \\
 \cdots & \xrightarrow{\quad} & a'_2 & \xrightarrow{\quad} & a'_1 & \xrightarrow{\quad} & a'_0, \\
 & & m'_3 & & m'_2 & & m'_1
 \end{array}$$

there exists a commutative square

$$\begin{array}{ccc}
 & a_0 & \\
 \nearrow & & \searrow f_0 \\
 \cdot & & b_0 \\
 \searrow & & \nearrow f'_0 \\
 & a'_0 &
 \end{array}$$

in \mathbb{A} , and

S2 for every serially commutative diagram

$$\begin{array}{ccccccc}
 & & m_3 & & m_2 & & m_1 \\
 \cdots & \xrightarrow{\quad} & a_2 & \xrightarrow{\quad} & a_1 & \xrightarrow{\quad} & a_0 \\
 & & \downarrow f_2 & \downarrow f'_2 & \downarrow f_1 & \downarrow f'_1 & \downarrow f_0 & \downarrow f'_0 \\
 \cdots & \xrightarrow{\quad} & b_2 & \xrightarrow{\quad} & b_1 & \xrightarrow{\quad} & b_0, \\
 & & p_3 & & p_2 & & p_1
 \end{array}$$

there exists a fork $\cdot \longrightarrow a_0 \begin{smallmatrix} \xrightarrow{f_0} \\ \xrightarrow{f'_0} \end{smallmatrix} b_0$ in \mathbb{A} .

In **S2**, ‘serially commutative’ means that $f_{n-1}m_n = p_nf_n$ and $f'_{n-1}m_n = p_nf'_n$ for all $n \geq 1$.

Example 6.1 For any small category \mathbb{A} there is a module $M : \mathbb{A} \multimap \mathbb{A}$ defined by $M(b, a) = \mathbb{A}(b, a)$, and (\mathbb{A}, M) is an equational system as long as $\sum_b \mathbb{A}(b, a)$ is finite for each $a \in \mathbb{A}$. Condition **S** says that \mathbb{A} is componentwise cofiltered; so, for instance, the equational system obtained by taking $\mathbb{A} = (0 \rightrightarrows 1)$ has no

universal solution. If \mathbb{A} is componentwise cofiltered then the universal solution is the functor $\mathbb{A} \longrightarrow \mathbf{Top}$ constant at the one-point space, with its unique coalgebra structure.

We now construct the universal solutions in **Set** and in **Top** of any equational system satisfying **S**. The proofs that they are indeed universal solutions are in §7 and §8, respectively.

The universal solution in Set

Let (\mathbb{A}, M) be an equational system. For each $a \in \mathbb{A}$, we have the category $\mathcal{J}(a)$ of complexes ending at a (§5). Each map $f : a \longrightarrow a'$ in \mathbb{A} induces a functor $\mathcal{J}f : \mathcal{J}(a) \longrightarrow \mathcal{J}(a')$, sending a complex

$$(a_\bullet, m_\bullet) = \left(\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0 = a \right)$$

to the complex

$$\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{fm_1} a'.$$

This defines a functor $\mathcal{J} : \mathbb{A} \longrightarrow \mathbf{Cat}$.

Write $\Pi_0 : \mathbf{Cat} \longrightarrow \mathbf{Set}$ for the functor sending a small category to its set of connected-components, and put $I = \Pi_0 \circ \mathcal{J} : \mathbb{A} \longrightarrow \mathbf{Set}$. We write $[a_\bullet, m_\bullet] \in I(a)$ for the connected-component of a complex $(a_\bullet, m_\bullet) \in \mathcal{J}(a)$. In §7 we will show that if (\mathbb{A}, M) satisfies **S** then I is nondegenerate.

Later we will analyze in detail the relation of connectedness in $\mathcal{J}(a)$, that is, equality in $I(a)$. For now, let us just note the following: for any diagram

$$\cdots \xrightarrow{m'_{n+2}} a'_{n+1} \xrightarrow{m'_{n+1}} a'_n \xrightarrow{f} a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0 = a,$$

there is a map

$$\begin{array}{ccccccc} \cdots & \xrightarrow{m'_{n+2}} & a'_{n+1} & \xrightarrow{m'_{n+1}} & a'_n & \xrightarrow{m_n f} & a_{n-1} \xrightarrow{m_{n-1}} \cdots \xrightarrow{m_1} a_0 = a \\ & & \downarrow 1 & & \downarrow f & & \downarrow 1 \\ \cdots & \xrightarrow{m'_{n+2}} & a'_{n+1} & \xrightarrow{fm'_{n+1}} & a_n & \xrightarrow{m_n} & a_{n-1} \xrightarrow{m_{n-1}} \cdots \xrightarrow{m_1} a_0 = a \end{array} \quad (24)$$

in $\mathcal{J}(a)$, so the complexes in the top and bottom rows represent the same element of $I(a)$.

Warning 6.2 The set $I(a)$ is *not* in general the limit of finite approximations. That is, let $\mathcal{J}_n(a)$ be the category whose objects are diagrams of the form

$$a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0 = a \quad (25)$$

and whose arrows are commutative diagrams; $\mathcal{J}_n(a)$ is finite, since M is. Let $I_n(a) = \Pi_0 \mathcal{J}_n(a)$. Then $\mathcal{J}(a)$ is the limit of the categories $\mathcal{J}_n(a)$, but $I(a)$ is typically not the limit of the sets $I_n(a)$. (An exception is when \mathbb{A} is discrete.) More precisely, the canonical map $I(a) \longrightarrow \lim_{\longleftarrow n} I_n(a)$ need not be injective,

since there may be two complexes in different connected-components of $\mathcal{J}(a)$ whose images in each $\mathcal{J}_n(a)$ are, nevertheless, always in the same component. An example is given at the end of 6.4, which also shows that the sequential limit of connected categories need not be connected.

The point can be clarified using the notion of ‘distance’ in a category. For objects A and A' of a category \mathcal{C} , the **distance** $d_{\mathcal{C}}(A, A')$ is the smallest number $n \in \mathbb{N}$ for which there exists a diagram

$$\begin{array}{ccccccc} & B_1 & & B_2 & & & B_n \\ & \swarrow & \searrow & \swarrow & \searrow & & \swarrow \searrow \\ A = A_0 & & A_1 & & \dots & & A_n = A' \end{array}$$

in \mathcal{C} , or ∞ if no such diagram exists. Thus, A and A' are in the same connected-component if and only if $d_{\mathcal{C}}(A, A') < \infty$. Any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ induces a distance-decreasing map: $d_{\mathcal{D}}(F(A), F(A')) \leq d_{\mathcal{C}}(A, A')$.

Now take $a \in \mathbb{A}$ and two complexes $\alpha, \alpha' \in \mathcal{J}(a)$. Writing $\text{pr}_n : \mathcal{J}(a) \longrightarrow \mathcal{J}_n(a)$ for projection, we have

$$d_{\mathcal{J}_n(a)}(\text{pr}_n(\alpha), \text{pr}_n(\alpha')) \leq d_{\mathcal{J}(a)}(\alpha, \alpha')$$

for all n . So if α and α' are in the same connected-component of $\mathcal{J}(a)$ then not only are the distances $d_{\mathcal{J}_n(a)}(\text{pr}_n(\alpha), \text{pr}_n(\alpha'))$ finite individually, but also there is an overall bound:

$$\sup_{n \geq 1} \left(d_{\mathcal{J}_n(a)}(\text{pr}_n(\alpha), \text{pr}_n(\alpha')) \right) < \infty. \quad (26)$$

Hence, the condition that $\text{pr}_n(\alpha)$ and $\text{pr}_n(\alpha')$ always represent the same element of $\mathcal{J}_n(a)$ is not enough to guarantee that α and α' represent the same element of $\mathcal{J}(a)$: (26) must also hold. (In fact, (26) is also a *sufficient* condition: Proposition 7.3).

We need to define a coalgebra structure on I , that is, a natural transformation $I \longrightarrow M \otimes I$. In order to do so, we first define one on $\text{ob } \mathcal{J}$, the composite of $\mathcal{J} : \mathbb{A} \longrightarrow \mathbf{Cat}$ with the objects functor $\text{ob} : \mathbf{Cat} \longrightarrow \mathbf{Set}$. The functor $\text{ob } \mathcal{J}$ is nondegenerate (whether or not **S** holds), since

$$\text{ob } \mathcal{J} \cong \sum_b \text{ob } \mathcal{J}(b) \times M(b, -) \quad (27)$$

and the class of nondegenerate functors is closed under sums (Theorem 4.11). The coalgebra structure $\iota : \text{ob } \mathcal{J} \longrightarrow M \otimes \text{ob } \mathcal{J}$ is defined by

$$\iota_a(\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a) = (a_1 \xrightarrow{m_1} a) \otimes (\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1)$$

($a \in \mathbb{A}$), or equivalently by taking ι_a to be the composite

$$\text{ob } \mathcal{J}(a) \xrightarrow{\sim} \sum_b M(b, a) \times \text{ob } \mathcal{J}(b) \xrightarrow{\text{quotient map}} (M \otimes \text{ob } \mathcal{J})(a). \quad (28)$$

We also have a quotient map $\pi : \text{ob } \mathcal{J} \longrightarrow I$, mapping a complex $(a_{\bullet}, m_{\bullet}) \in \text{ob } \mathcal{J}(a)$ to its connected-component $[a_{\bullet}, m_{\bullet}] \in I(a)$. It is easy to show that

the coalgebra structure on $\text{ob } \mathcal{J}$ induces a coalgebra structure on I , unique such that π is a map of coalgebras. We call this coalgebra structure ι , too; it is characterized by

$$\iota_a \left([\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a] \right) = (a_1 \xrightarrow{m_1} a) \otimes [\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1].$$

The universal solution in **Top**

Next we equip I with a topology. For each $a \in \mathbb{A}$, $n \in \mathbb{N}$, and truncated complex (25), there is a subset V_{m_1, \dots, m_n} of $I(a)$ consisting of all those $t \in I(a)$ such that

$$t = [\cdots \xrightarrow{m_{n+2}} a_{n+1} \xrightarrow{m_{n+1}} a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0 = a]$$

for some $m_{n+1}, a_{n+1}, m_{n+2}, \dots$. Equivalently, each sector $m : b \rightarrow a$ induces a function $\phi_m : I(b) \rightarrow I(a)$, with

$$\phi_m \left([\cdots \xrightarrow{p_2} b_1 \xrightarrow{p_1} b] \right) = [\cdots \xrightarrow{p_2} b_1 \xrightarrow{p_1} b \xrightarrow{m} a],$$

and then

$$V_{m_1, \dots, m_n} = \phi_{m_1} \phi_{m_2} \cdots \phi_{m_n}(I(a_n)).$$

Generate a topology on $I(a)$ by taking each such subset to be closed.

In order for (I, ι) to be a coalgebra in **Top**, the maps ι_a must be continuous (for every $a \in \mathbb{A}$) and the maps $I f$ must be continuous and closed (for every map f in \mathbb{A}). We will prove in §8 that these statements are true if **S** holds. In fact, it will follow from Lemma 9.1 that we have just given the sets $I(a)$ the coarsest possible topology for which (I, ι) is a coalgebra in **Top**.

Example 6.3 (Discrete systems) Let (\mathbb{A}, M) be a discrete equational system. Condition **S** holds trivially. For each $a \in \mathbb{A}$, the category $\mathcal{J}(a)$ is discrete, so $I(a)$ is simply $\text{ob } \mathcal{J}(a)$, the set of complexes ending at a . The topology on $I(a)$ is generated by declaring that for each diagram (25), the set of all complexes ending in (25) is closed in $I(a)$. This is the profinite topology on $I(a)$ defined at the end of §1.

Example 6.4 (Interval) We run through the constructions of this section in the case of the Freyd equational system (\mathbb{A}, M) (§2).

Condition **S** is easily verified. Theorem 2.1 states—although we have yet to prove it—that the universal solution is the coalgebra (I, ι) defined in §2; in particular, $I(1) = [0, 1]$. So an element of $[0, 1]$ should be an equivalence class of complexes

$$\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} 1.$$

If each a_n is 1 then each m_n is either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ and the complex is essentially a binary expansion; for instance, the diagram

$$\cdots \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1$$

corresponds to $0.101010\dots$, representing $2/3 \in [0, 1]$. Otherwise, the diagram is of the form

$$\dots \xrightarrow{\text{id}} 0 \xrightarrow{\text{id}} 0 \xrightarrow{m_{n+1}} 1 \xrightarrow{m_n} \dots \xrightarrow{m_1} 1$$

where $m_1, \dots, m_n \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ and $m_{n+1} \in \{0, \frac{1}{2}, 1\}$. Take, for instance,

$$\dots \xrightarrow{\text{id}} 0 \xrightarrow{\text{id}} 0 \xrightarrow{\frac{1}{2}} 1 \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[0, \frac{1}{2}]} 1.$$

To see which element t of $[0, 1]$ this represents, we can reason as follows. The $[0, \frac{1}{2}]$ says that $t \in [0, 1/2]$. The right-hand instance of $[\frac{1}{2}, 1]$ says that t is in the upper half of $[0, 1/2]$, that is, in $[1/4, 1/2]$. The left-hand instance of $[\frac{1}{2}, 1]$ says that t is in the upper half of $[1/4, 1/2]$, that is, in $[3/8, 1/2]$. The $\frac{1}{2}$ says that t is the midpoint of $[3/8, 1/2]$; that is, $t = 7/16$.

An element of $[0, 1]$ has at most two binary expansions, but may have infinitely many representations in $\mathcal{J}(1)$. For instance, the representations of $1/2$ are

$$\dots \xrightarrow{\text{id}} 0 \xrightarrow{\text{id}} 0 \xrightarrow{\frac{1}{2}} 1, \quad (29)$$

$$\dots \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1, \quad (30)$$

$$\dots \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[0, \frac{1}{2}]} 1, \quad (31)$$

and for any $n \in \mathbb{N}$,

$$\dots \xrightarrow{\text{id}} 0 \xrightarrow{1} 1 \xrightarrow{[\frac{1}{2}, 1]} \dots \xrightarrow{[\frac{1}{2}, 1]} 1 \xrightarrow{[0, \frac{1}{2}]} 1, \quad (32)$$

$$\dots \xrightarrow{\text{id}} 0 \xrightarrow{0} 1 \xrightarrow{[0, \frac{1}{2}]} \dots \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[\frac{1}{2}, 1]} 1 \quad (33)$$

with n copies of $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$ respectively. (Example 7.14 shows that, in a sense, (29) is the canonical representation.)

The construction says that two objects of $\mathcal{J}(1)$ represent the same element of $[0, 1]$ if and only if they are in the same connected-component. So, for instance, each of (29)–(33) are in the same component. The connected diagram

$$\begin{array}{ccccc} \dots & \xrightarrow{[\frac{1}{2}, 1]} & 1 & \xrightarrow{[\frac{1}{2}, 1]} & 1 & \xrightarrow{[0, \frac{1}{2}]} & 1 \\ & & \uparrow \tau & & \uparrow \tau & & \uparrow 1 \\ \dots & \xrightarrow{\text{id}} & 0 & \xrightarrow{\text{id}} & 0 & \xrightarrow{\frac{1}{2}} & 1 \\ & & \downarrow \sigma & & \downarrow \sigma & & \downarrow 1 \\ \dots & \xrightarrow{[0, \frac{1}{2}]} & 1 & \xrightarrow{[0, \frac{1}{2}]} & 1 & \xrightarrow{[\frac{1}{2}, 1]} & 1 \end{array}$$

shows that (29)–(31) are; the others are left to the reader.

The constructed topology on $[0, 1]$ is generated by taking as closed all subsets of the form $[k/2^n, l/2^n]$ where $k, l, n \in \mathbb{N}$ and $l \in \{k, k+1\}$. This is exactly the Euclidean topology.

Finally, this example shows that $I(a)$ need not be the limit of the finite approximations $I_n(a)$ (Warning 6.2). It is not hard to show that for each n , the category $\mathcal{J}_n(1)$ is connected, and so each $I_n(1)$ is a one-element set. But $I(1) = [0, 1]$, so $I(1) \not\cong \lim_{\leftarrow n} I_n(1)$.

7 Set-theoretic proofs

The main result of this section is that, for an equational system satisfying the solvability condition **S**, the construction above really does give the universal solution in **Set**.

We do not even know yet that this construction gives a coalgebra. Given an equational system (\mathbb{A}, M) , we do have a functor $I : \mathbb{A} \longrightarrow \mathbf{Set}$ and a natural transformation $\iota : I \longrightarrow M \otimes I$, but in order for (I, ι) to be called an M -coalgebra, it must, by definition, be nondegenerate. A large part of this section is devoted to proving that. (The proof requires condition **S**.) It then follows quite quickly that (I, ι) is the universal solution.

An element of one of the sets $I(a)$ is an equivalence class of complexes. We finish the section by showing that under very mild conditions on \mathbb{A} , each such element has a canonical complex representing it.

Connectedness in $\mathcal{J}(a)$

The functor $I : \mathbb{A} \longrightarrow \mathbf{Set}$ was constructed by a two-step process: first we defined $\mathcal{J} : \mathbb{A} \longrightarrow \mathbf{Cat}$, then we took $I(a)$ to be the set of connected-components of $\mathcal{J}(a)$. To understand I we therefore need to understand the relation of connectedness in the category $\mathcal{J}(a)$. We now begin to analyze this relation. This analysis is what gives the theory much of its substance, and we will return to it later too (7.7, 8.1).

Notation: if Γ is the limit of a diagram

$$\cdots \longrightarrow \Gamma_3 \longrightarrow \Gamma_2 \longrightarrow \Gamma_1 \quad (34)$$

in some category, pr_n will denote both the projection $\Gamma \longrightarrow \Gamma_n$ and the given map $\Gamma_m \longrightarrow \Gamma_n$ for any $m \geq n$.

Lemma 7.1 (König [Kön]) *The limit in **Set** of a diagram (34) of finite nonempty sets is nonempty. More precisely, for any sequence $(x_n)_{n \geq 1}$ with $x_n \in \Gamma_n$, there exists an element $y \in \lim_{\leftarrow n} \Gamma_n$ such that*

$$\forall r \geq 1, \exists n \geq r : \text{pr}_r(x_n) = \text{pr}_r(y).$$

Proof Take a sequence $(x_n)_{n \geq 1}$ with $x_n \in \Gamma_n$. We define, for each $r \geq 1$, an infinite subset \mathcal{N}_r of \mathbb{N}^+ and an element $y_r \in \Gamma_r$ such that

- for all $r \geq 1$, $\mathcal{N}_r \subseteq \mathcal{N}_{r-1} \cap \{r, r+1, \dots\}$ (writing $\mathcal{N}_0 = \mathbb{N}^+$)
- for all $r \geq 1$ and $n \in \mathcal{N}_r$, $\text{pr}_r(x_n) = y_r$.

Suppose inductively that $r \geq 1$ and \mathcal{N}_{r-1} is defined. As n runs through the infinite set $\mathcal{N}_{r-1} \cap \{r, r+1, \dots\}$, $\text{pr}_r(x_n)$ takes values in the finite set Γ_r , so takes some value $y_r \in \Gamma_r$ infinitely often. Putting

$$\mathcal{N}_r = \{n \in \mathcal{N}_{r-1} \cap \{r, r+1, \dots\} \mid \text{pr}_r(x_n) = y_r\}$$

completes the induction.

For each $r \geq 1$ we have $y_r = \text{pr}_r(y_{r+1})$, since we may choose $n \in \mathcal{N}_{r+1}$, and then

$$\text{pr}_r(y_{r+1}) = \text{pr}_r(\text{pr}_{r+1}(x_n)) = \text{pr}_r(x_n) = y_r.$$

So there is a unique element $y \in \lim_{\leftarrow n} \Gamma_n$ such that $\text{pr}_r(y) = y_r$ for all $r \geq 1$. Given $r \geq 1$, we may choose $n \in \mathcal{N}_r$, and then $n \geq r$ and $\text{pr}_r(x_n) = y_r = \text{pr}_r(y)$, as required. \square

We now use the notion of distance in a category, introduced in Warning 6.2.

Lemma 7.2 (Distance in a limit) *Let*

$$\cdots \longrightarrow \mathbb{L}_3 \longrightarrow \mathbb{L}_2 \longrightarrow \mathbb{L}_1 \quad (35)$$

be a diagram of finite categories, and let A, A' be objects of $\mathbb{L} = \lim_{\leftarrow n} \mathbb{L}_n$. Then

$$d_{\mathbb{L}}(A, A') = \sup_{n \geq 1} \left(d_{\mathbb{L}_n}(\text{pr}_n(A), \text{pr}_n(A')) \right).$$

Proof Write $s = \sup_{n \geq 1} d_{\mathbb{L}_n}(\text{pr}_n(A), \text{pr}_n(A'))$. Certainly $d_{\mathbb{L}}(A, A') \geq s$, since functors are distance-decreasing (Warning 6.2). Now let us show that $d_{\mathbb{L}}(A, A') \leq s$. Certainly this is true if $s = \infty$; assume that $s < \infty$. For each $n \in \mathbb{N}$, let Γ_n be the set of diagrams

$$\begin{array}{ccccccc} & & \beta_1 & & \beta_2 & & \beta_s \\ & \swarrow & & \searrow & \swarrow & \searrow & \\ \text{pr}_n(A) = \alpha_0 & & & \alpha_1 & & \cdots & \alpha_s = \text{pr}_n(A') \end{array}$$

in \mathbb{L}_n . Then Γ_n is finite since \mathbb{L}_n is, and nonempty by hypothesis. So by König's Lemma, $\lim_{\leftarrow n} \Gamma_n$ is nonempty; that is, $d_{\mathbb{L}}(A, A') \leq s$. \square

Proposition 7.3 (Equality and distance) *Let \mathbb{A} be a small category and $M : \mathbb{A} \rightarrow \mathbb{A}$ a finite module. Let $a \in \mathbb{A}$ and $(a_\bullet, m_\bullet), (a'_\bullet, m'_\bullet) \in \mathcal{J}(a)$. Then*

$$\begin{aligned} [a_\bullet, m_\bullet] &= [a'_\bullet, m'_\bullet] \\ \iff \sup_{n \geq 1} \left(d_{\mathcal{J}_n(a)}(a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0, a'_n \xrightarrow{m'_n} \cdots \xrightarrow{m'_1} a'_0) \right) < \infty. \end{aligned}$$

Proof Since M is finite, each category $\mathcal{J}_n(a)$ is finite. Now apply Lemma 7.2 with $\mathbb{L}_n = \mathcal{J}_n(a)$. \square

This result gives a criterion for connectedness in the category $\mathcal{J}(a)$ of complexes, purely in terms of the categories $\mathcal{J}_n(a)$ of *truncated* complexes.

I is nondegenerate

We begin with a standard categorical construction. Any functor $\mathcal{X} : \mathbb{B} \rightarrow \mathbf{Cat}$ has a **category of elements** $\mathbb{E}(\mathcal{X})$. An object of $\mathbb{E}(\mathcal{X})$ is a pair (b, x) with $b \in \mathbb{B}$ and $x \in \mathcal{X}(b)$, and an arrow $(b, x) \rightarrow (b', x')$ is a pair (g, ξ) with $g : b \rightarrow b'$ in \mathbb{B} and $\xi : (\mathcal{X}g)(x) \rightarrow x'$ in $\mathcal{X}(b')$. This is related to the notion of the category of elements of a **Set**-valued functor $X : \mathbb{B} \rightarrow \mathbf{Set}$ (defined

before 2.6) by the isomorphism $\mathbb{E}(X) \cong \mathbb{E}(D \circ X)$, where $D : \mathbf{Set} \longrightarrow \mathbf{Cat}$ is the functor assigning to each set the corresponding discrete category.

As remarked after Corollary 4.12, a \mathbf{Set} -valued functor X is nondegenerate if and only if $\mathbb{E}(X)$ is componentwise cofiltered. We now show that the \mathbf{Cat} -valued functor \mathcal{J} has a kind of nondegeneracy property: $\mathbb{E}(\mathcal{J})$ is componentwise cofiltered.

For the rest of this section, let (\mathbb{A}, M) be an equational system satisfying the solvability condition **S**.

The category of elements $\mathbb{E}(\mathcal{J})$ of $\mathcal{J} : \mathbb{A} \longrightarrow \mathbf{Cat}$ is the category of complexes. For each $n \in \mathbb{N}$ we have a functor $\mathcal{J}_n : \mathbb{A} \longrightarrow \mathbf{Cat}$ (as in Warning 6.2); its category of elements is the category of complexes of length n . Then $\mathbb{E}(\mathcal{J})$ is the limit in \mathbf{Cat} of

$$\cdots \longrightarrow \mathbb{E}(\mathcal{J}_2) \longrightarrow \mathbb{E}(\mathcal{J}_1).$$

Proposition 7.4 *$\mathbb{E}(\mathcal{J})$ is componentwise cofiltered.*

Proof We have to prove that every diagram $\cdot \longrightarrow \cdot \longleftarrow \cdot$ in $\mathbb{E}(\mathcal{J})$ can be completed to a commutative square, and that every parallel pair $\cdot \rightrightarrows \cdot$ can be completed to a fork. The two cases are very similar, so I will just do the first.

Take a diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{m_3} & a_2 & \xrightarrow{m_2} & a_1 & \xrightarrow{m_1} & a_0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \cdots & \xrightarrow{p_3} & b_2 & \xrightarrow{p_2} & b_1 & \xrightarrow{p_1} & b_0 \\ & & \uparrow f'_2 & & \uparrow f'_1 & & \uparrow f'_0 \\ \cdots & \xrightarrow{m'_3} & a'_2 & \xrightarrow{m'_2} & a'_1 & \xrightarrow{m'_1} & a'_0 \end{array} \quad (36)$$

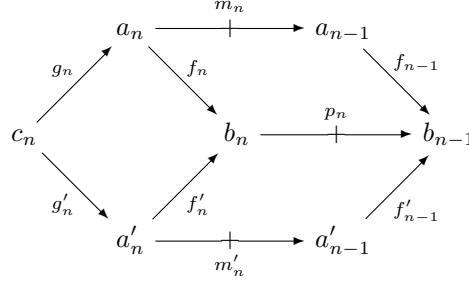
of shape $(\cdot \longrightarrow \cdot \longleftarrow \cdot)$ in $\mathbb{E}(\mathcal{J})$. For $n \geq 1$, let Γ_n be the set of diagrams

$$\begin{array}{ccccccc} & & m_n & & m_{n-1} & & \cdots & & m_2 & & \\ a_n & \xrightarrow{m_n} & a_{n-1} & \xrightarrow{m_{n-1}} & \cdots & \xrightarrow{m_2} & a_1 & & & & \\ g_n \uparrow & & g_{n-1} \uparrow & & & & g_1 \uparrow & & & & \\ c_n & \xrightarrow{q_n} & c_{n-1} & \xrightarrow{q_{n-1}} & \cdots & \xrightarrow{q_2} & c_1 & & & & \\ g'_n \downarrow & & g'_{n-1} \downarrow & & & & g'_1 \downarrow & & & & \\ a'_n & \xrightarrow{m'_n} & a'_{n-1} & \xrightarrow{m'_{n-1}} & \cdots & \xrightarrow{m'_2} & a'_1 & & & & \end{array} \quad (37)$$

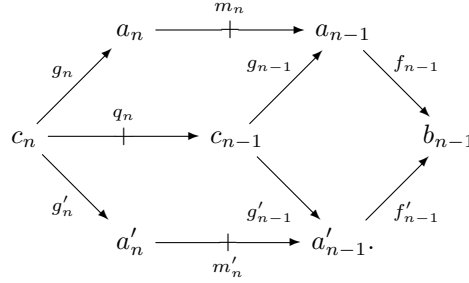
satisfying $f_1 g_1 = f'_1 g'_1$, \dots , $f_n g_n = f'_n g'_n$. There are evident projections $\Gamma_{n+1} \longrightarrow \Gamma_n$. We will apply König's Lemma.

Each set Γ_n is finite, by finiteness of M and the fact that the indexing in (37) starts at 1, not 0. I claim that Γ_n is also nonempty. Indeed, **S1** implies that

there exist c_n , g_n , and g'_n making



commute, and then nondegeneracy of M (condition **ND1** at the end of §4) implies that the outside of this diagram can also be filled in as



Repeating this argument $(n - 2)$ times gives an element of Γ_n , as required.

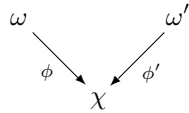
By König's Lemma, $\lim_{\leftarrow n} \Gamma_n$ is nonempty; that is, diagram (36) with its rightmost column removed can be completed to a commutative square in $\mathbb{E}(\mathcal{J})$. Using the diagram-filling argument one more time shows that (36) can be, too. \square

In the next few results we see that for general reasons, $\mathbb{E}(\mathcal{J})$ being componentwise cofiltered has two consequences: each category $\mathcal{J}(a)$ is also componentwise cofiltered, and $I : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate.

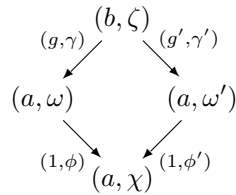
Lemma 7.5 *Let $\mathcal{J} : \mathbb{B} \longrightarrow \mathbf{Cat}$ be a functor on a small category \mathbb{B} . If $\mathbb{E}(\mathcal{J})$ is componentwise cofiltered then $\mathcal{J}(a)$ is componentwise cofiltered for each $a \in \mathbb{B}$.*

Proof We have to prove that every diagram $\cdot \longrightarrow \cdot \longleftarrow \cdot$ in $\mathcal{J}(a)$ can be completed to a commutative square, and that every parallel pair $\cdot \rightrightarrows \cdot$ can be completed to a fork. Again I will just do the first case; the second is similar.

Take a diagram



in $\mathcal{J}(a)$. Then there is a commutative square



in $\mathbb{E}(\mathcal{J})$. Commutativity says that $g = g'$ and that the square

$$\begin{array}{ccc} & (\mathcal{J}g)(\zeta) & \\ \gamma \swarrow & & \searrow \gamma' \\ \omega & & \omega' \\ \phi \searrow & & \swarrow \phi' \\ & \chi & \end{array}$$

in $\mathcal{J}(a)$ commutes, as required. \square

Proposition 7.6 $\mathcal{J}(a)$ is componentwise cofiltered for each $a \in \mathbb{A}$. \square

We will repeatedly use the following corollary.

Corollary 7.7 (Equality and spans) Let $a \in \mathbb{A}$, and let $(a_\bullet, m_\bullet), (a'_\bullet, m'_\bullet) \in \mathcal{J}(a)$. Then $[a_\bullet, m_\bullet] = [a'_\bullet, m'_\bullet]$ if and only if there exists a span

$$(a_\bullet, m_\bullet) \longleftarrow \cdot \longrightarrow (a'_\bullet, m'_\bullet)$$

in $\mathcal{J}(a)$.

Proof By Proposition 7.6, $\mathcal{J}(a)$ has the square-completion property; then use Lemma 4.8. \square

Lemma 7.8 Let $\mathcal{J} : \mathbb{B} \longrightarrow \mathbf{Cat}$ be a functor on a small category \mathbb{B} . If $\mathbb{E}(\mathcal{J})$ is componentwise cofiltered then so is $\mathbb{E}(\Pi_0 \circ \mathcal{J})$.

Proof Once again the proof splits into two similar cases. For variety I will do the second: that every diagram $(a, [\omega]) \xrightarrow[f']{f} (b, [\chi])$ in $\mathbb{E}(\Pi_0 \circ \mathcal{J})$ extends to a fork.

Since $[(\mathcal{J}f)(\omega)] = [\chi] = [(\mathcal{J}f')(\omega)]$, Lemmas 4.8 and 7.5 imply that there exists a span

$$\begin{array}{ccc} & \xi & \\ \delta \swarrow & & \searrow \delta' \\ (\mathcal{J}f)(\omega) & & (\mathcal{J}f')(\omega) \end{array}$$

in $\mathcal{J}(b)$. We therefore have a finite connected diagram (solid arrows)

$$\begin{array}{ccc} & (c, \zeta) & \\ \text{---} (g, \gamma) \text{---} \swarrow & & \searrow \text{---} (fg=f'g, \theta) \text{---} \\ (a, \omega) & & (b, \xi) \\ \downarrow (f, 1) & \swarrow (f', 1) \quad \searrow (1, \delta) & \downarrow (1, \delta') \\ (b, (\mathcal{J}f)(\omega)) & & (b, (\mathcal{J}f')(\omega)) \end{array}$$

in $\mathbb{E}(\mathcal{J})$, so by hypothesis there exists a dotted commutative diagram, giving a fork

$$(c, [\zeta]) \xrightarrow{g} (a, [\omega]) \xrightarrow[f']{f} (b, [\chi])$$

in $\mathbb{E}(\Pi_0 \circ \mathcal{J})$. \square

Proposition 7.9 (Nondegeneracy) $I : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate.

Proof By Proposition 7.4 and Lemma 7.8, $\mathbb{E}(I)$ is componentwise cofiltered. By the remark after Corollary 4.12, this is equivalent to I being nondegenerate. \square

Hence (I, ι) is an M -coalgebra. By Lambek's Lemma, a necessary condition for it to be the universal solution is that ι is an isomorphism. We can prove this fact directly—and we need to, since it will be used in the proof that (I, ι) is the universal solution.

Corollary 7.10 (Fixed point) $\iota : I \longrightarrow M \otimes I$ is an isomorphism.

Proof It is enough to show that $\iota_a : I(a) \longrightarrow (M \otimes I)(a)$ is bijective for each $a \in \mathbb{A}$. Certainly ι_a is surjective. For injectivity, suppose that

$$\iota_a([\cdots \xrightarrow{m_2} a_1 \xrightarrow{m_1} a]) = \iota_a([\cdots \xrightarrow{m'_2} a'_1 \xrightarrow{m'_1} a]),$$

that is,

$$m_1 \otimes [\cdots \xrightarrow{m_2} a_1] = m'_1 \otimes [\cdots \xrightarrow{m'_2} a'_1].$$

By nondegeneracy of I and Lemma 5.1, there exist a commutative square

$$\begin{array}{ccc} & b & \\ f \swarrow & & \searrow f' \\ a_1 & & a'_1 \\ m_1 \searrow & & \swarrow m'_1 \\ & a & \end{array}$$

and an element $[\cdots \xrightarrow{p_2} b_1 \xrightarrow{p_1} b] \in I(b)$ such that

$$\begin{aligned} [\cdots \xrightarrow{p_2} b_1 \xrightarrow{fp_1} a_1] &= [\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1], \\ [\cdots \xrightarrow{p_2} b_1 \xrightarrow{f'p_1} a'_1] &= [\cdots \xrightarrow{m'_3} a'_2 \xrightarrow{m'_2} a'_1]. \end{aligned}$$

Then

$$\begin{aligned} [\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a] &= [\cdots \xrightarrow{p_2} b_1 \xrightarrow{fp_1} a_1 \xrightarrow{m_1} a] \\ &= [\cdots \xrightarrow{p_2} b_1 \xrightarrow{p_1} b \xrightarrow{m_1 f} a], \end{aligned}$$

using the observation at (24) (§6). But $m_1 f = m'_1 f'$, so by symmetry of argument,

$$[\cdots \xrightarrow{m_2} a_1 \xrightarrow{m_1} a] = [\cdots \xrightarrow{m'_2} a'_1 \xrightarrow{m'_1} a],$$

as required. \square

Existence Given $a \in \mathbb{A}$ and $x \in X(a)$, we may choose a resolution $(a_\bullet, m_\bullet, x_\bullet)$ of x and put

$$\bar{\xi}_a(x) = [a_\bullet, m_\bullet] \in I(a).$$

This defines for each a a function $\bar{\xi}_a : X(a) \longrightarrow I(a)$, which by Corollary 5.3 is independent of choice of resolution. The maps $(\bar{\xi}_a)_{a \in \mathbb{A}}$ define a natural transformation $\bar{\xi} : X \longrightarrow I$; that is, if $a \xrightarrow{f} a'$ is a map in \mathbb{A} and $x \in X(a)$ then $\bar{\xi}_{a'}(fx) = f\bar{\xi}_a(x)$. For choose a resolution $(a_\bullet, m_\bullet, x_\bullet)$ of x : then

$$((\cdots \xrightarrow{m_2} a_1 \xrightarrow{fm_1} a'), (fx, x_1, x_2, \dots))$$

is a resolution of fx , so

$$\bar{\xi}_{a'}(fx) = [\cdots \xrightarrow{m_2} a_1 \xrightarrow{fm_1} a'] = f[\cdots \xrightarrow{m_2} a_1 \xrightarrow{m_1} a] = f\bar{\xi}_a(x).$$

Moreover, $\bar{\xi}$ is a map of coalgebras; that is, if $a \in \mathbb{A}$ and $x \in X(a)$ then

$$(M \otimes \bar{\xi})_a \xi_a(x) = \iota_a \bar{\xi}_a(x).$$

For choose a resolution $(a_\bullet, m_\bullet, x_\bullet)$ of x : then

$$((\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1), (x_1, x_2, x_3, \dots))$$

is a resolution of x_1 , so

$$\begin{aligned} (M \otimes \bar{\xi})_a \xi_a(x) &= (M \otimes \bar{\xi})_a(m_1 \otimes x_1) = m_1 \otimes \bar{\xi}_{a_1}(x_1) \\ &= m_1 \otimes [\cdots \xrightarrow{m_2} a_1] = \iota_a \bar{\xi}_a(x). \end{aligned}$$

Uniqueness Let $\tilde{\xi} : (X, \xi) \longrightarrow (I, \iota)$ be a map of coalgebras, $a \in \mathbb{A}$, and $x \in X(a)$. We show that $\tilde{\xi}_a(x) = \bar{\xi}_a(x)$.

Choose a resolution $(a_\bullet, m_\bullet, x_\bullet)$ of x , and for each $n \in \mathbb{N}$, write

$$\tilde{\xi}_{a_n}(x_n) = [\cdots \xrightarrow{m_n^2} a_n^1 \xrightarrow{m_n^1} a_n^0 = a_n].$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned} (M \otimes \tilde{\xi})_{a_n} \xi_{a_n}(x_n) &= (M \otimes \tilde{\xi})_{a_n}(m_{n+1} \otimes x_{n+1}) \\ &= m_{n+1} \otimes \tilde{\xi}_{a_{n+1}}(x_{n+1}) \\ &= m_{n+1} \otimes [\cdots \xrightarrow{m_{n+1}^2} a_{n+1}^1 \xrightarrow{m_{n+1}^1} a_{n+1}^0 = a_{n+1}] \\ &= \iota_{a_n} \left([\cdots \xrightarrow{m_{n+1}^1} a_{n+1}^0 = a_{n+1} \xrightarrow{m_{n+1}} a_n] \right). \end{aligned}$$

On the other hand,

$$(M \otimes \tilde{\xi})_{a_n} \xi_{a_n}(x_n) = \iota_{a_n} \tilde{\xi}_{a_n}(x_n)$$

since $\tilde{\xi}$ is a map of coalgebras. Since ι_{a_n} is injective (Corollary 7.10),

$$\begin{aligned} [\cdots \xrightarrow{m_{n+1}^1} a_{n+1}^0 = a_{n+1} \xrightarrow{m_{n+1}} a_n] &= \tilde{\xi}_{a_n}(x_n) \\ &= [\cdots \xrightarrow{m_n^2} a_n^1 \xrightarrow{m_n^1} a_n^0 = a_n] \end{aligned}$$

for each $n \in \mathbb{N}$. So Proposition 7.11 applies, and

$$[\cdots \xrightarrow{m_0^2} a_0^1 \xrightarrow{m_0^1} a_0^0] = [\cdots \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0] \in I(a),$$

that is, $\tilde{\xi}_a(x) = \bar{\xi}_a(x)$, as required. \square

The canonical representation of an element of the universal solution

An element of the universal solution is an equivalence class of complexes. One might not expect every element to have a *canonical* complex representing it, since, for example, not every real number has a *canonical* decimal expansion. So it is perhaps a surprise that under very mild conditions on \mathbb{A} , satisfied in every example of an equational system in this paper, every element of the universal solution does indeed have a canonical representing complex.

This result was suggested to me by André Joyal, who has kindly allowed me to include it here. No later results depend on it.

The main theorem is:

Theorem 7.13 (Canonical representation) *Suppose that \mathbb{A} is Cauchy-complete. Let $a \in \mathbb{A}$. Then each connected-component of $\mathcal{I}(a)$ has an initial object.*

(Recall our standing assumption for this section that (\mathbb{A}, M) is an equational system satisfying **S**.)

Example 7.14 (Interval) In Example 6.4 we considered the Freyd system (\mathbb{A}, M) and the various representations of real numbers in $[0, 1]$. We saw that $1/2 \in [0, 1]$ is represented by infinitely many complexes ((29)–(33)); that is, the connected-component of $\mathcal{I}(1)$ corresponding to $1/2 \in I(1)$ has infinitely many objects. Its initial object is the complex (29), which can therefore be regarded as the *canonical* representation of $1/2$.

We now prepare to prove Theorem 7.13.

Lemma 7.15 *Nondegenerate functors preserve finite connected limits.*

Proof Let \mathbb{B} be a small category and $X : \mathbb{B} \longrightarrow \mathbf{Set}$ a nondegenerate functor. We have

$$X \cong \left(\mathbb{B} \xrightarrow{\text{Yoneda}} [\mathbb{B}^{\text{op}}, \mathbf{Set}] \xrightarrow{- \otimes X} \mathbf{Set} \right)$$

and the Yoneda embedding preserves limits. \square

Lemma 7.16 (Connected limits of truncated complexes) *Let \mathbb{K} be a finite connected category. If \mathbb{A} has limits of shape \mathbb{K} then so does $\mathcal{I}_n(a)$, for every $a \in \mathbb{A}$ and $n \in \mathbb{N}$.*

Remark The same proof shows that under the same hypotheses, $\mathcal{I}(a)$ has limits of shape \mathbb{K} . The projections $\mathcal{I}(a) \longrightarrow \mathcal{I}_n(a)$ and $\mathcal{I}_n(a) \longrightarrow \mathbb{A}$ all preserve those limits.

Proof Suppose that \mathbb{A} has limits of shape \mathbb{K} , let $a \in \mathbb{A}$, and let $n \in \mathbb{N}$. Let $D : \mathbb{K} \longrightarrow \mathcal{J}_n(a)$ be a diagram, and write its value at an object $k \in \mathbb{K}$ as

$$D(k) = (a_n^k \xrightarrow{m_n^k} \cdots \xrightarrow{m_1^k} a_0^k = a).$$

We construct a limit cone on D .

For each $r \in \{1, \dots, n\}$, the diagram

$$\begin{array}{ccc} D_r : \mathbb{K} & \longrightarrow & \mathbb{A}, \\ k & \longmapsto & a_r^k \end{array}$$

has a limit cone $(a_r \xrightarrow{p_r^k} a_r^k)_{k \in \mathbb{K}}$. There is also a trivial limit cone $(a_0 \xrightarrow{p_0^k} a_0^k)_{k \in \mathbb{K}}$ with $a_0 = a$ and $p_0^k = 1_a$. By Lemma 7.15, the functor $M(a_r, -)$ preserves limits of shape \mathbb{K} for each r ; hence

$$\left(M(a_r, a_{r-1}) \xrightarrow{p_{r-1}^k} M(a_r, a_{r-1}^k) \right)_{k \in \mathbb{K}}$$

is a limit cone. It follows that, for each r , there is a unique sector $m_r : a_r \dashrightarrow a_{r-1}$ such that $p_{r-1}^k m_r = m_r^k p_r^k$ for all $k \in \mathbb{K}$. This gives a cone

$$\left(\begin{array}{ccccccc} a_n & \xrightarrow{m_n} & a_{n-1} & \xrightarrow{m_{n-1}} & \cdots & \xrightarrow{m_1} & a_0 = a \\ p_n^k \downarrow & & \downarrow p_{n-1}^k & & & & \downarrow p_0^k = 1_a \\ a_n^k & \xrightarrow{m_n^k} & a_{n-1}^k & \xrightarrow{m_{n-1}^k} & \cdots & \xrightarrow{m_1^k} & a_0^k = a \end{array} \right)_{k \in \mathbb{K}}$$

on D , and it is straightforward to check, using Lemma 7.15 again, that it is a limit cone. \square

Let \mathbb{H} be the two-element monoid consisting of the identity and an idempotent. A diagram of shape \mathbb{H} in a category \mathcal{C} is an idempotent in \mathcal{C} , and a limit—or equally, a colimit—of such a diagram is a splitting of the idempotent. So Lemma 7.16 implies:

Corollary 7.17 *Suppose that \mathbb{A} is Cauchy-complete. Then $\mathcal{J}_n(a)$ is Cauchy-complete for every $a \in \mathbb{A}$ and $n \in \mathbb{N}$.* \square

We will use the fact that the filtered cocompletion of a small category \mathbb{B} is $\mathbf{Flat}(\mathbb{B}^{\text{op}}, \mathbf{Set})$, the full subcategory of $[\mathbb{B}^{\text{op}}, \mathbf{Set}]$ consisting of the flat functors (§6.3, 6.5 of [Bor]). In particular, $\mathbf{Flat}(\mathbb{B}^{\text{op}}, \mathbf{Set})$ has filtered colimits, and any filtered colimit preserved by the Yoneda embedding $\mathbf{y} : \mathbb{B} \longrightarrow \mathbf{Flat}(\mathbb{B}^{\text{op}}, \mathbf{Set})$ is **absolute**, that is, preserved by every functor out of \mathbb{B} .

Lemma 7.18 (Finite Cauchy-complete categories) *Let \mathbb{B} be a finite category. Then*

\mathbb{B} is Cauchy-complete $\iff \mathbb{B}$ has filtered colimits $\iff \mathbb{B}$ has cofiltered limits.

In that case, filtered colimits and cofiltered limits in \mathbb{B} are absolute.

Proof By duality, we need only consider filtered colimits. Since the category \mathbb{H} is filtered, a category with filtered colimits is always Cauchy-complete. Conversely, suppose that \mathbb{B} is Cauchy-complete. By finiteness and Proposition 4.15(a), the functor $\mathbf{y} : \mathbb{B} \longrightarrow \mathbf{Flat}(\mathbb{B}^{\text{op}}, \mathbf{Set})$ is an equivalence; so by the remarks above, \mathbb{B} has filtered colimits and they are absolute. \square

Proposition 7.19 (Cofiltered limits of truncated complexes) *Suppose that \mathbb{A} is Cauchy-complete. Then for each $n \in \mathbb{N}$ and $a \in \mathbb{A}$, the category $\mathcal{J}_n(a)$ has filtered colimits and cofiltered limits, and they are absolute.*

Proof Follows from Corollary 7.17, Lemma 7.18, and finiteness of $\mathcal{J}_n(a)$. \square

We now use these results about truncated complexes to deduce results about ordinary, non-truncated, complexes.

Lemma 7.20 *Let \mathbb{K} be a category and let $a \in \mathbb{A}$. Suppose that for all $n \in \mathbb{N}$, the category $\mathcal{J}_n(a)$ has limits of shape \mathbb{K} and the projection functor $\text{pr}_n : \mathcal{J}_{n+1}(a) \longrightarrow \mathcal{J}_n(a)$ preserves them. Then $\mathcal{J}(a)$ has limits of shape \mathbb{K} .*

Remark This almost follows from the fact that $\mathcal{J}(a) = \varprojlim_n \mathcal{J}_n(a)$. However, this is a *strict* (1-categorical) limit, whereas the functors pr_n only preserve limits in the usual sense that a certain canonical map is an *isomorphism*. One can, for instance, write down a sequence (35) of categories and functors in which each of the categories has a terminal object and each of the functors preserves them, but the limit does not have a terminal object. Something extra is therefore needed in order to build limits in $\mathcal{J}(a)$.

Proof First observe that each functor pr_n is an **isofibration** [JT, Lack]: given an object $\alpha \in \mathcal{J}_{n+1}(a)$ and an isomorphism $j : \text{pr}_n(\alpha) \longrightarrow \beta$ in $\mathcal{J}_n(a)$, there exists an isomorphism $i : \alpha \longrightarrow \alpha'$ such that $\text{pr}_n(\alpha') = \beta$ and $\text{pr}_n(i) = j$.

Now take a diagram $D : \mathbb{K} \longrightarrow \mathcal{J}(a)$. Write $D_n : \mathbb{K} \longrightarrow \mathcal{J}_n(a)$ for the composite of D with the projection $\mathcal{J}(a) \longrightarrow \mathcal{J}_n(a)$. We may choose a limit cone on D_1 ; then, using the isofibration property, a limit cone on D_2 whose image in $\mathcal{J}_1(a)$ is the chosen cone on D_1 ; and so on. This compatible sequence of cones defines a cone on D itself, which is a limit cone. \square

Proposition 7.19 and Lemma 7.20 together imply:

Proposition 7.21 (Cofiltered limits of complexes) *Suppose that \mathbb{A} is Cauchy-complete. Then for all $a \in \mathbb{A}$, the category $\mathcal{J}(a)$ has cofiltered limits.* \square

We can now prove that every element of $I(a)$ has a canonical complex representing it.

Proof of Theorem 7.13 Let $a \in \mathbb{A}$ and let \mathbb{K} be a connected-component of $\mathcal{J}(a)$. Condition **S** implies that $\mathcal{J}(a)$ is componentwise cofiltered (Proposition 7.6), so \mathbb{K} is cofiltered. Then by Proposition 7.21, the inclusion $\mathbb{K} \hookrightarrow \mathcal{J}(a)$ has a limit. But since \mathbb{K} is a connected-component, a limit cone on this inclusion functor amounts to a limit cone on the identity functor $\mathbb{K} \longrightarrow \mathbb{K}$, which by Lemma X.1 of [Mac] amounts to an initial object of \mathbb{K} . \square

8 Topological proofs

Fix an equational system (\mathbb{A}, M) satisfying the solvability condition **S**. In this section we show that (I, ι) , with the topology defined in §6, is an M -coalgebra in **Top**, and indeed the universal solution in **Top**. Along the way we prove that each space $I(a)$ is compact Hausdorff.

$I(a)$ is Hausdorff

We begin with the Hausdorff property. Recall the sets V_{m_1, \dots, m_n} defined in §6. Define, for each $n \in \mathbb{N}$ and $a \in \mathbb{A}$, a binary relation R_n^a on $I(a)$ by

$$\begin{aligned} R_n^a &= \bigcup \{ V_{p_1, \dots, p_n} \times V_{p_1, \dots, p_n} \mid (b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0) \in \mathcal{J}_n(a) \} \\ &\subseteq I(a) \times I(a). \end{aligned}$$

Equivalently, $(t, t') \in R_n^a$ when there exists $(b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0) \in \mathcal{J}_n(a)$ such that t and t' can both be written in the form

$$[\cdots \xrightarrow{\quad} \cdot \xrightarrow{\quad} b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0].$$

As a subset of $I(a) \times I(a)$, R_n^a is closed, by finiteness of $\mathcal{J}_n(a)$. As a relation, R_n^a is reflexive and symmetric, but not in general transitive: for example, in the Freyd system, $R_1^1 = [0, \frac{1}{2}]^2 \cup [\frac{1}{2}, 1]^2 \subseteq [0, 1]^2$.

Given a set S , write Δ_S for the diagonal $\{(s, s) \mid s \in S\} \subseteq S \times S$.

Proposition 8.1 (Relations determine equality) *For each $a \in \mathbb{A}$, we have $\bigcap_{n \in \mathbb{N}} R_n^a = \Delta_{I(a)}$.*

Proof Certainly $\bigcap_{n \in \mathbb{N}} R_n^a \supseteq \Delta_{I(a)}$. Conversely, let $(t, t') \in \bigcap_{n \in \mathbb{N}} R_n^a$, writing

$$\begin{aligned} t &= [\cdots \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0 = a], \\ t' &= [\cdots \xrightarrow{m'_2} a'_1 \xrightarrow{m'_1} a'_0 = a]. \end{aligned}$$

For each $n \in \mathbb{N}$, we may choose $(b_n^n \xrightarrow{p_n^n} \cdots \xrightarrow{p_n^1} b_n^0) \in \mathcal{J}_n(a)$ such that $t, t' \in V_{p_n^1, \dots, p_n^n}$. By Corollary 7.7, there is for each $n \in \mathbb{N}$ a span in $\mathcal{J}(a)$ of the form

$$\begin{array}{ccccccc} \cdots & \xrightarrow{m_{n+2}} & a_{n+1} & \xrightarrow{m_{n+1}} & a_n & \xrightarrow{m_n} & \cdots & \xrightarrow{m_1} & a_0 = a \\ & & \uparrow & & \uparrow & & & & \uparrow 1_a \\ \cdots & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & a \\ & & \downarrow & & \downarrow & & & & \downarrow 1_a \\ \cdots & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & b_n^n & \xrightarrow{p_n^n} & \cdots & \xrightarrow{p_n^1} & b_n^0 = a. \end{array}$$

Applying the projection functor $\mathcal{J}(a) \longrightarrow \mathcal{J}_n(a)$, we have

$$d_{\mathcal{J}_n(a)}(a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0, b_n^n \xrightarrow{p_n^n} \cdots \xrightarrow{p_n^1} b_n^0) \leq 1.$$

The same is true for t' , so by the triangle inequality,

$$d_{\mathcal{J}_n(a)}(a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0, a'_n \xrightarrow{m'_n} \cdots \xrightarrow{m'_1} a'_0) \leq 2$$

for each $n \in \mathbb{N}$. So by Proposition 7.3, $t = t'$. \square

Corollary 8.2 (Hausdorff) *For each $a \in \mathbb{A}$, the space $I(a)$ is Hausdorff.*

Proof $\Delta_{I(a)}$ is closed in $I(a) \times I(a)$, being the intersection of the closed subsets R_n^a . \square

Corollary 8.3 (Singletons) *Let $a \in \mathbb{A}$ and $(a_\bullet, m_\bullet) \in \mathcal{J}(a)$. Then $\bigcap_{r \in \mathbb{N}} V_{m_1, \dots, m_r} = \{(a_\bullet, m_\bullet)\}$.*

Proof By Proposition 8.1, the left-hand side has at most one element; but clearly $[a_\bullet, m_\bullet]$ is an element. \square

(I, ι) is a topological coalgebra

By definition, an element of the set $I(a)$ is an equivalence class of elements of $\text{ob } \mathcal{J}(a)$, and the coalgebra structure on I is induced by the coalgebra structure on $\text{ob } \mathcal{J}$ via the quotient map $\pi : \text{ob } \mathcal{J} \longrightarrow I$. The next phase of the proof is to show that, in a similar sense, (I, ι) is a quotient of $(\text{ob } \mathcal{J}, \iota)$ as a coalgebra in **Top**.

For this to make sense, we need to put a topology on $\text{ob } \mathcal{J}$. For each $a \in \mathbb{A}$, the set $\text{ob } \mathcal{J}(a)$ is the limit of the diagram of finite sets

$$\cdots \longrightarrow \text{ob } \mathcal{J}_2(a) \longrightarrow \text{ob } \mathcal{J}_1(a).$$

Equipping each set $\text{ob } \mathcal{J}_n(a)$ with the discrete topology and taking the limit in **Top** gives a topology on $\text{ob } \mathcal{J}(a)$ (the profinite topology). We used the same construction in §1: writing $\text{ob } \mathbb{A}$ for the discrete category with the same objects as \mathbb{A} , there is an evident discrete equational system $(\text{ob } \mathbb{A}, M)$, and its universal solution is $(\text{ob } \mathcal{J}, \iota)$.

In this way, $\text{ob } \mathcal{J}$ becomes a functor $\mathbb{A} \longrightarrow \mathbf{Top}$. Each space $\text{ob } \mathcal{J}(a)$ is compact Hausdorff. Hence, recalling from (27) that the **Set**-valued functor $\text{ob } \mathcal{J}$ is nondegenerate, the **Top**-valued functor $\text{ob } \mathcal{J}$ is nondegenerate. The maps ι_a are continuous, since in (28) the first map is a homeomorphism and the second is a topological quotient map. So $(\text{ob } \mathcal{J}, \iota)$ is a coalgebra in **Top**.

We will show that for each a , the map $\pi_a : \text{ob } \mathcal{J}(a) \longrightarrow I(a)$ exhibits $I(a)$ as a topological quotient of $\text{ob } \mathcal{J}(a)$. From that we will deduce that (I, ι) too is a coalgebra in **Top**.

Lemma 8.4 (Membership of basic closed sets) *Let $a \in \mathbb{A}$, $n \in \mathbb{N}$, $(a_\bullet, m_\bullet) \in \mathcal{J}(a)$, and $(b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0) \in \mathcal{J}_n(a)$. Then*

$$[a_\bullet, m_\bullet] \in V_{p_1, \dots, p_n} \iff \text{for all } r \in \mathbb{N}, V_{m_1, \dots, m_r} \cap V_{p_1, \dots, p_n} \neq \emptyset.$$

Proof ' \Rightarrow ' is trivial. For ' \Leftarrow ', we may choose for each $r \in \mathbb{N}$ complexes

$$\begin{aligned} \alpha_r &= (\cdots \xrightarrow{m_{r+2}^r} a_{r+1}^r \xrightarrow{m_{r+1}^r} a_r \xrightarrow{m_r} \cdots \xrightarrow{m_1} a_0 = a), \\ \beta_r &= (\cdots \xrightarrow{p_{n+2}^r} b_{n+1}^r \xrightarrow{p_{n+1}^r} b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0 = a) \end{aligned}$$

such that $[\alpha_r] = [\beta_r]$. By Corollary 7.7, there is for each $r \in \mathbb{N}$ a span

$$\alpha_r \longleftarrow \cdot \longrightarrow \beta_r$$

in $\mathcal{J}(a)$. Applying König's Lemma (7.1) to the limit $\text{ob } \mathcal{J}(a) = \lim_{\longleftarrow r} \text{ob } \mathcal{J}_r(a)$ and the elements $\text{pr}_r(\beta_r) \in \text{ob } \mathcal{J}_r(a)$ gives a complex $\beta \in \mathcal{J}(a)$ with the following property:

for all $r \in \mathbb{N}$, there exists $k \geq r$ such that $\text{pr}_r(\beta) = \text{pr}_r(\beta_k)$.

Taking $r = n$ gives $\text{pr}_n(\beta) = (b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0)$. Hence $[\beta] \in V_{p_1, \dots, p_n}$.

I claim that $[a_\bullet, m_\bullet] = [\beta]$; the result will follow. Indeed, let $r \in \mathbb{N}$. Choose $k \geq r$ such that $\text{pr}_r(\beta) = \text{pr}_r(\beta_k)$. We have $d_{\mathcal{J}(a)}(\alpha_k, \beta_k) \leq 1$, so, applying $\text{pr}_r : \mathcal{J}(a) \longrightarrow \mathcal{J}_r(a)$,

$$d_{\mathcal{J}_r(a)}(\text{pr}_r(a_\bullet, m_\bullet), \text{pr}_r(\beta)) = d_{\mathcal{J}_r(a)}(\text{pr}_r(\alpha_k), \text{pr}_r(\beta_k)) \leq 1.$$

So by Proposition 7.3, $[a_\bullet, m_\bullet] = [\beta]$, as required. \square

Proposition 8.5 (Topological quotient) *For each $a \in \mathbb{A}$, the canonical surjection $\pi_a : \text{ob } \mathcal{J}(a) \longrightarrow I(a)$ is a topological quotient map.*

Proof First, π_a is continuous. Let $n \in \mathbb{N}$ and $(b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0) \in \mathcal{J}_n(a)$; we must show that $\pi_a^{-1}V_{p_1, \dots, p_n}$ is a closed subset of $\text{ob } \mathcal{J}(a)$. By Lemma 8.4,

$$\pi_a^{-1}V_{p_1, \dots, p_n} = \bigcap_{r \in \mathbb{N}} \text{pr}_r^{-1}W_r$$

where

$$W_r = \{(a_r \xrightarrow{m_r} \cdots \xrightarrow{m_1} a_0) \in \text{ob } \mathcal{J}_r(a) \mid V_{m_1, \dots, m_r} \cap V_{p_1, \dots, p_n} \neq \emptyset\}.$$

But each space $\text{ob } \mathcal{J}_r(a)$ is discrete and each map pr_r is continuous, so $\bigcap_{r \in \mathbb{N}} \text{pr}_r^{-1}W_r$ is closed, as required.

Since $\text{ob } \mathcal{J}(a)$ is compact and $I(a)$ is Hausdorff, π_a is closed. So π_a is a continuous closed surjection, and therefore a quotient map. \square

Corollary 8.6 (Compactness) *For each $a \in \mathbb{A}$, the space $I(a)$ is compact.* \square

Corollary 8.7 (Topological coalgebra) *(I, ι) is an M -coalgebra in **Top**.*

Proof First we have to show that for each map $f : a \longrightarrow a'$ in \mathbb{A} , the map $If : I(a) \longrightarrow I(a')$ is continuous and closed. There is a commutative square

$$\begin{array}{ccc} \text{ob } \mathcal{J}(a) & \xrightarrow{\pi_a} & I(a) \\ \text{ob } \mathcal{J}f \downarrow & & \downarrow If \\ \text{ob } \mathcal{J}(a') & \xrightarrow{\pi_{a'}} & I(a') \end{array}$$

in which π_a is a topological quotient map and $\text{ob } \mathcal{J}(f)$ and $\pi_{a'}$ are continuous, so If is also continuous. But $I(a)$ is compact and $I(a')$ Hausdorff, so If is closed.

We also have to show that for each $a \in \mathbb{A}$, the map $\iota_a : I(a) \longrightarrow (M \otimes I)(a)$ is continuous. This is proved by a similar argument, using the square

$$\begin{array}{ccc} \text{ob } \mathcal{J}(a) & \xrightarrow{\pi_a} & I(a) \\ \downarrow \iota_a & & \downarrow \iota_a \\ (M \otimes \text{ob } \mathcal{J})(a) & \xrightarrow{(M \otimes \pi)_a} & (M \otimes I)(a). \end{array}$$

□

I is the terminal **Top**-coalgebra

Our final task is to prove that for any M -coalgebra (X, ξ) in **Top**, the unique map $\bar{\xi} : (X, \xi) \longrightarrow (I, \iota)$ of coalgebras in **Set** is continuous. To do this we show that the inverse image of each basic closed set V_{p_1, \dots, p_n} is closed, where $n \in \mathbb{N}$ and $b_n \xrightarrow{p_n} \dots \xrightarrow{p_1} b_0 = a$.

Some care is needed in describing this inverse image. Given an element $x \in X(a)$, the complexes along which x can be resolved all lie in the same connected-component of $\mathcal{J}(a)$, namely $\bar{\xi}_a(x)$. However, there may be complexes in this component along which x cannot be resolved. So if we write

$$V_{p_1, \dots, p_n}^X \subseteq X(a)$$

for the set of elements of $X(a)$ that can be resolved along some complex of the form

$$\dots \xrightarrow{\quad} \cdot \xrightarrow{\quad} b_n \xrightarrow{p_n} \dots \xrightarrow{p_1} b_0 = a,$$

then

$$V_{p_1, \dots, p_n}^X \subseteq \bar{\xi}_a^{-1} V_{p_1, \dots, p_n} \quad (39)$$

but the inclusion may be strict. The following example illustrates this.

Example 8.8 Let (\mathbb{A}, M) be the Freyd system. Choose an endpoint-preserving continuous map $\xi_1 : [0, 1] \longrightarrow [0, 2]$ such that $\xi_1(2/3) = 2/3$; this defines an M -coalgebra structure ξ on $X = (\{\star\} \xrightleftharpoons[1]{0} [0, 1])$. The element $2/3 \in X(1)$ has a unique resolution, which is along the complex

$$\dots \xrightarrow{[0, \frac{1}{2}]} 1 \xrightarrow{[0, \frac{1}{2}]} 1.$$

Hence

$$\bar{\xi}_1(2/3) = 0 = [\dots \xrightarrow{\text{id}} 0 \xrightarrow{\text{id}} 0 \xrightarrow{0} 1] \in V_{p_1},$$

where $p_1 = 0 : 0 \xrightarrow{\quad} 1$. So $2/3 \in \bar{\xi}_1^{-1} V_{p_1}$, even though $2/3$ cannot be resolved along any complex ending in p_1 .

(The notation V_{p_1, \dots, p_n}^X is explained by the fact that $V_{p_1, \dots, p_n}^I = V_{p_1, \dots, p_n}$. This follows from (39) and the existence of canonical resolutions (§7).)

Lemma 8.9 (Inverse image of basic closed sets) *Let $a \in \mathbb{A}$, $n \in \mathbb{N}$, and $(b_n \xrightarrow{p_n} \cdots \xrightarrow{p_1} b_0) \in \mathcal{I}_n(a)$. Let (X, ξ) be an M -coalgebra in **Set**. Then*

$$\bar{\xi}_a^{-1} V_{p_1, \dots, p_n} = \bigcap_{r \in \mathbb{N}} \bigcup V_{m_1, \dots, m_r}^X \quad (40)$$

where the union is over all $(a_r \xrightarrow{m_r} \cdots \xrightarrow{m_1} a_0) \in \mathcal{I}_r(a)$ such that

$$V_{m_1, \dots, m_r} \cap V_{p_1, \dots, p_n} \neq \emptyset. \quad (41)$$

Proof Let $x \in \bar{\xi}_a^{-1} V_{p_1, \dots, p_n}$, and choose a complex (a_\bullet, m_\bullet) along which x can be resolved. Then for all r ,

$$[a_\bullet, m_\bullet] = \bar{\xi}_a(x) \in V_{m_1, \dots, m_r} \cap V_{p_1, \dots, p_n},$$

and in particular (41) holds. Also $x \in V_{m_1, \dots, m_r}^X$ by definition, so x is in the right-hand side of (40).

Conversely, let x be an element of the right-hand side of (40). By König's Lemma (7.1), we may choose a complex $(a_\bullet, m_\bullet) \in \mathcal{I}(a)$ such that for all r , (41) holds and $x \in V_{m_1, \dots, m_r}^X$. By Lemma 8.4, $[a_\bullet, m_\bullet] \in V_{p_1, \dots, p_n}$. Now using (39) and Corollary 8.3,

$$x \in \bigcap_{r \in \mathbb{N}} V_{m_1, \dots, m_r}^X \subseteq \bar{\xi}_a^{-1} \bigcap_{r \in \mathbb{N}} V_{m_1, \dots, m_r} = \bar{\xi}_a^{-1} \{[a_\bullet, m_\bullet]\} \subseteq \bar{\xi}_a^{-1} V_{p_1, \dots, p_n},$$

as required. \square

This describes the inverse images of the basic closed sets. We now prepare to show that they are closed.

Lemma 8.10 *Let (X, ξ) be an M -coalgebra in **Top**, $r \in \mathbb{N}$, and $(a_r \xrightarrow{m_r} \cdots \xrightarrow{m_1} a_0) \in \mathcal{I}_r(a)$. Then V_{m_1, \dots, m_r}^X is a closed subset of $X(a)$.*

Proof When $r = 0$ this is trivial. Suppose inductively that the result holds for $r \in \mathbb{N}$, and let $(a_{r+1} \xrightarrow{m_{r+1}} \cdots \xrightarrow{m_1} a_0) \in \mathcal{I}_{r+1}(a)$. We have

$$V_{m_1, \dots, m_{r+1}}^X = \xi_a^{-1} \left(m_1 \otimes V_{m_2, \dots, m_{r+1}}^X \right)$$

where $m_1 \otimes S$ means the image of a subset $S \subseteq X(a_1)$ under the map

$$m_1 \otimes - : X(a_1) \longrightarrow (M \otimes X)(a).$$

But $V_{m_2, \dots, m_{r+1}}^X$ is closed by inductive hypothesis, $m_1 \otimes -$ is closed by Corollary 5.6, and ξ_a is continuous, so $V_{m_1, \dots, m_{r+1}}^X$ is closed in $X(a)$. \square

Theorem 8.11 (Universal solution in Top) *(I, ι) is the universal solution of (\mathbb{A}, M) in **Top**.*

Proof Let (X, ξ) be an M -coalgebra in **Top**. It remains to show that for each $a \in \mathbb{A}$, the map $\bar{\xi}_a : X(a) \longrightarrow I(a)$ is continuous, and this follows from Lemmas 8.9 and 8.10. \square

9 Recognizing the universal solution

We have seen that an equational system possesses a universal solution if and only if an explicit condition **S** holds; if so, the universal solution is unique and can be constructed explicitly.

Few examples have been given so far. In principle one can take any equational system (\mathbb{A}, M) satisfying **S** and find the universal solution (I, ι) by going through the explicit construction. In practice this is cumbersome and it is much quicker to apply one of the Recognition Theorems proved below, as follows.

We might sometimes observe that some familiar space has a recursive decomposition, and we might ask whether it can be characterized as the universal solution of some equational system (or rather, one of the spaces $I(a)$ of which the universal solution is made up). The Recognition Theorems provide a way to confirm such guesses.

For example, we might note that the standard topological simplices Δ^n admit barycentric subdivision, which exhibits each simplex as a gluing-together of smaller simplices. This barycentric subdivision can be expressed as an isomorphism $\Delta^\bullet \cong M \otimes \Delta^\bullet$, where M is a certain module and Δ^\bullet is the functor $n \mapsto \Delta^n$. Using one of the Recognition Theorems, we can confirm that Δ^\bullet is in fact the universal solution of M (Example 10.12), thus giving a new characterization of the spaces Δ^n .

We prove two results. The Precise Recognition Theorem gives necessary and sufficient conditions for a fixed point of an equational system to be a universal solution. The Crude Recognition Theorem gives merely sufficient conditions, but they are very quick to check and satisfied in many examples of interest. These two theorems will be applied in §10 to yield examples of universal solutions, and in Appendix C to determine exactly which spaces are recursively realizable.

We begin by listing some of the properties enjoyed by (I, ι) , the universal solution constructed in §6. These will form the basis of the Precise Recognition Theorem.

From now up to and including Lemma 9.3, fix an equational system (\mathbb{A}, M) satisfying the solvability condition **S**.

The first property of (I, ι) is that it is a fixed point of M , that is, an M -coalgebra whose structure map is an isomorphism. (Recall that M -coalgebras, and in particular fixed points, are nondegenerate by definition.) A fixed point (J, γ) is a coalgebra, but can also be regarded as an algebra (J, ψ) where $\psi = \gamma^{-1}$. By definition, an M -**algebra** (in **Top**) is a nondegenerate functor $J : \mathbb{A} \longrightarrow \mathbf{Top}$ together with a map $\psi : M \otimes J \longrightarrow J$. By the universal property of $M \otimes J$ (Appendix A), ψ amounts to a family

$$\left(J(b) \xrightarrow{\psi_m} J(a) \right)_{b \multimap a}^m$$

of continuous maps ψ_m , indexed over all sectors $m : b \multimap a$, satisfying a naturality axiom: $\psi_{fmg} = (Jf) \circ \psi_m \circ (Jg)$ whenever m is a sector and f and g are arrows in \mathbb{A} for which this makes sense.

For example, the fixed point (I, ι) has algebra structure $\phi = \iota^{-1}$, where the components ϕ_m are as defined in §6.

Lemma 9.1 *Let $(J, \gamma = \psi^{-1})$ be a fixed point of M in **Top**. Then for each sector $b \xrightarrow{m} a$, the map $J(b) \xrightarrow{\psi_m} J(a)$ is closed.*

Proof ψ_m is the composite

$$J(b) \xrightarrow{m \otimes -} (M \otimes J)(a) \xrightarrow[\sim]{\psi_a} J(a)$$

and $m \otimes -$ is closed by nondegeneracy of J and Corollary 5.6. \square

Being a fixed point alone is not enough to imply being the universal solution: for example, the constant functor \emptyset is always a fixed point and not usually the universal solution. A functor $J : \mathbb{A} \longrightarrow \mathbf{Set}$ is **occupied** if for all $a \in \mathbb{A}$,

$$\mathcal{I}(a) \neq \emptyset \implies J(a) \neq \emptyset.$$

When J has an M -coalgebra structure, being occupied means that the sets $J(a)$ are ‘not empty unless they have to be’: for if $\mathcal{I}(a)$ is empty then $J(a)$ must be empty, since any element of $J(a)$ would have a resolution $(a_\bullet, m_\bullet, x_\bullet)$ with $(a_\bullet, m_\bullet) \in \mathcal{I}(a)$. The second property enjoyed by I is that, trivially, it is occupied.

The third property of I is that the spaces $I(a)$ are metrizable:

Lemma 9.2 *A compact space is metrizable if and only if it is Hausdorff and has a countable basis of open sets.*

Proof See [Bou, IX.2.9] (where ‘compact’ means compact Hausdorff). \square

One naturally asks how a metric can be defined. There are many possible metrics and apparently no canonical choice among them, but the following result tells us all we need to know. Recall (from the beginning of §8) that for each $a \in \mathbb{A}$ and $n \in \mathbb{N}$ we have a closed binary relation R_n^a on $I(a)$, with $R_0^a \supseteq R_1^a \supseteq \dots$.

Lemma 9.3 (Metric on $I(a)$) *Let $a \in \mathbb{A}$ and let d be a metric on $I(a)$ compatible with its topology. Then for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$(t, t') \in R_n^a \implies d(t, t') < \varepsilon.$$

Proof Let $\varepsilon > 0$. Since $I(a)$ is compact, so too is $d^{-1}[\varepsilon, \infty)$, the inverse image of $[\varepsilon, \infty)$ under the continuous map $d : I(a) \times I(a) \longrightarrow [0, \infty)$.

By Proposition 8.1 we have $\bigcap_{n \in \mathbb{N}} R_n^a = \Delta_{I(a)}$, so $\bigcap_{n \in \mathbb{N}} R_n^a \cap d^{-1}[\varepsilon, \infty) = \emptyset$. But each subset R_n^a is closed, so by compactness, there is some $n \in \mathbb{N}$ for which $R_n^a \cap d^{-1}[\varepsilon, \infty) = \emptyset$. \square

To state the main theorem, we need a little more notation.

Given an equational system (\mathbb{A}, M) , a fixed point $(J, \gamma = \psi^{-1})$, and a truncated complex $a_n \xrightarrow{m_n} \dots \xrightarrow{m_1} a_0$, write

$$V_{m_1, \dots, m_n}^J = \psi_{m_1} \cdots \psi_{m_n} J(a_n),$$

the image of the composite map

$$J(a_n) \xrightarrow{\psi_{m_n}} \dots \xrightarrow{\psi_{m_1}} J(a_0).$$

Although we will not need to know it, this is the same as the set V_{m_1, \dots, m_n}^J defined in §8 for an arbitrary coalgebra J .

Write $\text{diam}(S)$ for the diameter of a metric space S .

Theorem 9.4 (Precise Recognition Theorem) *Let (\mathbb{A}, M) be an equational system. The following are equivalent conditions on a fixed point (J, γ) of M in **Top**:*

- a. (J, γ) is a universal solution of (\mathbb{A}, M) in **Top**
- b. J is occupied, and for each $a \in \mathbb{A}$ the space $J(a)$ is compact and can be metrized in such a way that

$$\inf_{n \in \mathbb{N}} \sup_{m_1, \dots, m_n} \text{diam}(V_{m_1, \dots, m_n}^J) = 0$$

where the supremum is over all

$$\left(a_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} a_0 = a \right) \in \mathcal{I}_n(a)$$

- c. J is occupied; for each $a \in \mathbb{A}$, the space $J(a)$ is compact; and for every complex (a_\bullet, m_\bullet) , the set $\bigcap_{n \in \mathbb{N}} V_{m_1, \dots, m_n}^J$ has at most one element.

The only part of the proof requiring substantial work is (c) \implies (a). We first prepare the ground.

Let (\mathbb{A}, M) be an equational system, (X, ξ) an M -coalgebra in **Set**, and (J, γ) a fixed point. Write $\psi = \gamma^{-1}$, as usual. A natural transformation $\omega : X \longrightarrow J$ is a map of coalgebras if and only if for all $a \in \mathbb{A}$, the square

$$\begin{array}{ccc} X(a) & \xrightarrow{\xi_a} & (M \otimes X)(a) \\ \omega_a \downarrow & & \downarrow (M \otimes \omega)_a \\ J(a) & \xrightarrow{\gamma_a} & (M \otimes J)(a) \end{array} \quad (42)$$

commutes. Let $x \in X(a)$. Writing

$$\xi_a(x) = \left(b \xrightarrow{m} a \right) \otimes y,$$

commutativity of the square at x says that $\gamma_a \omega_a(x) = m \otimes \omega_b(y)$, or equivalently, $\omega_a(x) = \psi_m \omega_b(y)$. Hence $\omega_a(x)$ lies in the subset

$$\bigcap \psi_m J(b)$$

of $J(a)$, where the intersection is over all $b \xrightarrow{m} a$ and $y \in X(b)$ such that $x = m \otimes y$. (Note that this subset is defined without reference to ω .) The same reasoning can be applied to each such y , further constraining where in $J(b)$ the element $\omega_b(y)$ can lie; and so on, iteratively. This suggests the following definition.

For each $a \in \mathbb{A}$ and $x \in X(a)$, define a sequence $(K_n(x))_{n \in \mathbb{N}}$ of subsets of $J(a)$ by

$$\begin{aligned} K_0(x) &= J(a), \\ K_{n+1}(x) &= \bigcap_{\xi(x)=m \otimes y} \psi_m K_n(y) \end{aligned}$$

where the intersection is over all $b \in \mathbb{A}$, $m \in M(b, a)$ and $y \in X(b)$ such that $\xi_a(x) = m \otimes y$. We will show that $(K_n(x))_{n \in \mathbb{N}}$ is a decreasing sequence of closed subsets of $J(a)$, and, moreover, that if (J, γ) is the universal solution then $\bigcap_n K_n(x)$ is the singleton set $\{\bar{\xi}_a(x)\}$, where $\bar{\xi}$ is the unique coalgebra map $X \longrightarrow J$. The sets $K_n(x)$ can therefore be thought of as approximations to $\bar{\xi}_a(x)$.

Example 9.5 Let (\mathbb{A}, M) be the Freyd equational system (§2) and let (J, γ) be its universal solution (I, ι) . Let (X, ξ) be the subcoalgebra of (I, ι) defined by taking $X(0) = \emptyset$ and

$$X(1) = \{x \in [0, 1] \mid x \text{ is not a dyadic rational}\}.$$

Then for each $x \in X(1)$ there is a unique pair (m, y) such that $\xi(x) = m \otimes y$, so the intersection in the definition of $K_{n+1}(x)$ is indexed over a one-element set. In fact, $K_n(x) \subseteq [0, 1]$ is the unique interval of the form $[r/2^n, (r+1)/2^n]$, with r an integer, containing x .

Lemma 9.6 *Let (\mathbb{A}, M) be an equational system, let (X, ξ) be an M -coalgebra, and let (J, ψ) be a fixed point of M . Then for all $a \in \mathbb{A}$ and $x \in X(a)$,*

- a. $K_0(x) \supseteq K_1(x) \supseteq \dots$
- b. $K_n(x)$ is closed in $J(a)$ for all $n \in \mathbb{N}$
- c. $(Jf)(K_n(x)) \subseteq K_n(fx)$ for all maps $f : a \longrightarrow a'$ in \mathbb{A} and $n \in \mathbb{N}$
- d. if $J(a)$ is compact and J is occupied then $\bigcap_{n \in \mathbb{N}} K_n(x) \neq \emptyset$.

Proof Part (a) is a straightforward induction, and part (b) follows from Lemma 9.1 by another induction.

Part (c) is also an induction. For $n = 0$ it is trivial. Suppose inductively that it holds for some $n \in \mathbb{N}$. Let $t \in K_{n+1}(x)$, and let $b' \xrightarrow{m'} a'$ and $y' \in X(b')$ with $\xi(fx) = m' \otimes y'$; we have to show that $ft \in \psi_{m'} K_n(y')$.

We may choose $b \xrightarrow{m} a$ and $y \in X(b)$ such that $\xi(x) = m \otimes y$. Then $m' \otimes y' = \xi(fx) = fm \otimes y$, so by Lemma 5.1 there exist a commutative square

$$\begin{array}{ccc} & c & \\ g \swarrow & & \searrow g' \\ b & & b' \\ fm \searrow & & \swarrow m' \\ & a' & \end{array}$$

and $z \in X(c)$ such that $y = gz$ and $y' = g'z$. Now

$$\xi(x) = m \otimes y = m \otimes gz = mg \otimes z,$$

so $t \in \psi_{mg} K_n(z)$ by definition of $K_n(z)$. Hence

$$\begin{aligned} ft &\in (Jf)\psi_{mg} K_n(z) = \psi_{fmg} K_n(z) = \psi_{m'g'} K_n(z) = \psi_{m'}(Jg') K_n(z) \\ &\subseteq \psi_{m'} K_n(g'z) = \psi_{m'} K_n(y') \end{aligned}$$

(the penultimate step by inductive hypothesis), as required.

Part (d) will follow from compactness and parts (a) and (b) once we know that each set $K_n(x)$ is nonempty. We prove this by induction on n over all $a \in \mathbb{A}$ and $x \in X(a)$ simultaneously.

For $n = 0$, let $a \in \mathbb{A}$ and $x \in X(a)$. There exists a resolution of x , and in particular an element of $\mathcal{I}(a)$. Since J is occupied, $\emptyset \neq J(a) = K_0(x)$.

Now let $n \in \mathbb{N}$, $a \in \mathbb{A}$, and $x \in X(a)$; we have to prove that $K_{n+1}(x) \neq \emptyset$. Since $K_{n+1}(x)$ is an intersection of a family of closed subsets of a compact space, it suffices to show that the intersection of any finite sub-family is nonempty. So, suppose that $r \in \mathbb{N}$ and $\xi(x) = m_1 \otimes y_1 = \cdots = m_r \otimes y_r$ where $b_i \xrightarrow{m_i} a$ and $y_i \in X(b_i)$; we have to show that

$$\bigcap_{i=1}^r \psi_{m_i} K_n(y_i) \neq \emptyset. \quad (43)$$

When $r = 0$ this says that $J(a) \neq \emptyset$, which we have just shown. Suppose that $r \geq 1$. By Lemma 5.1 and an easy induction on r , there exist $c \xrightarrow{p} a$, an element $z \in X(c)$ and maps $g_i : c \longrightarrow b_i$ such that $m_i g_i = p$ and $g_i z = y_i$ for all $i \in \{1, \dots, r\}$. Then $\xi(x) = p \otimes z$, and for each i ,

$$\psi_{m_i} K_n(y_i) = \psi_{m_i} K_n(g_i z) \supseteq \psi_{m_i} (J g_i) K_n(z) = \psi_{m_i g_i} K_n(z) = \psi_p K_n(z),$$

using (c). Hence $\bigcap_{i=1}^r \psi_{m_i} K_n(y_i) \supseteq \psi_p K_n(z)$. But $K_n(z) \neq \emptyset$ by inductive hypothesis, so $\bigcap_{i=1}^r \psi_{m_i} K_n(y_i) \neq \emptyset$, proving (43). \square

Proof of Theorem 9.4

(a) \implies (b) Assume (a). By Theorem B.1, condition **S** holds, so (J, γ) is the universal solution (I, ι) constructed in §6. Certainly I is occupied and each space $I(a)$ is compact. Lemmas 9.2 and 9.3 then give metrics with the property required.

(b) \implies (c) Trivial.

(c) \implies (a) Assume (c). First we show that (J, γ) is the universal solution in **Set**. So, let (X, ξ) be an M -coalgebra in **Set**; we construct a coalgebra map $(X, \xi) \longrightarrow (J, \gamma)$ and prove that it is the unique such.

For each $a \in \mathbb{A}$ and $x \in X(a)$ we have a sequence $(K_n(x))_{n \in \mathbb{N}}$ of subsets of $J(a)$, defined above. By Lemma 9.6(d), $\bigcap_{n \in \mathbb{N}} K_n(x)$ has at least one element. On the other hand, choose a resolution $(a_\bullet, m_\bullet, x_\bullet)$ of x . Then for each $n \in \mathbb{N}$, writing $\psi = \gamma^{-1}$, we have

$$K_n(x) \subseteq \psi_{m_1} \cdots \psi_{m_n} J(a_n) = V_{m_1, \dots, m_n}^J.$$

So by (c), $\bigcap_{n \in \mathbb{N}} K_n(x)$ has at most one element. Hence we may define, for each $a \in \mathbb{A}$, a function $\bar{\xi}_a : X(a) \longrightarrow J(a)$ by $\{\bar{\xi}_a(x)\} = \bigcap_{n \in \mathbb{N}} K_n(x)$.

The family $(\bar{\xi}_a)_{a \in \mathbb{A}}$ is a natural transformation $X \longrightarrow J$. Indeed, let $f : a \longrightarrow a'$ be a map in \mathbb{A} . Then for all $n \in \mathbb{N}$,

$$f \bar{\xi}_a(x) \in (Jf) K_n(x) \subseteq K_n(fx)$$

by Lemma 9.6(c), so $f\bar{\xi}_a(x) = \bar{\xi}_{a'}(fx)$, as required.

I claim that $\bar{\xi}$ is a map $(X, \xi) \longrightarrow (J, \gamma)$ of coalgebras in **Set**. Let $a \in \mathbb{A}$ and $x \in X(a)$, and write

$$\xi_a(x) = \left(b \xrightarrow{m} a \right) \otimes y.$$

Then by the observation at (42), we have to show that $\psi_m(\bar{\xi}_b(y)) \in K_n(x)$ for all $n \in \mathbb{N}$. When $n = 0$ this is certainly true. Now let $n \geq 1$; we have to show that for all m' and y' such that

$$\xi_a(x) = \left(b' \xrightarrow{m'} a \right) \otimes y',$$

we have $\psi_m \bar{\xi}_b(y) \in \psi_{m'} K_{n-1}(y')$. Since $m \otimes y = m' \otimes y'$, there exist a commutative square

$$\begin{array}{ccc} & c & \\ g \swarrow & & \searrow g' \\ b & & b' \\ m \searrow & & \swarrow m' \\ & a & \end{array}$$

and $z \in X(c)$ such that $gz = y$ and $g'z = y'$. Hence

$$\begin{aligned} \psi_m \bar{\xi}_b(y) &= \psi_m \bar{\xi}_b(gz) = \psi_m g \bar{\xi}_c(z) = \psi_{mg} \bar{\xi}_c(z) = \psi_{m'g'} \bar{\xi}_c(z) \\ &= \psi_{m'} \bar{\xi}_{b'}(y') \in \psi_{m'} K_{n-1}(y') \end{aligned}$$

(the last equality by symmetry), as required.

For uniqueness, let $\tilde{\xi} : (X, \xi) \longrightarrow (J, \gamma)$ be a map of coalgebras in **Set**. We show by induction on n that $\tilde{\xi}_a(x) \in K_n(x)$ for all $a \in \mathbb{A}$ and $x \in X(a)$; the result follows. For $n = 0$ this is trivial. Let $n \geq 1$, $a \in \mathbb{A}$, and $x \in X(a)$. If $\xi_a(x) = (b \xrightarrow{m} a) \otimes y$ then, as observed at (42), $\tilde{\xi}$ being a map of coalgebras implies that $\tilde{\xi}_a(x) = \psi_m \tilde{\xi}_b(y)$; so by inductive hypothesis, $\tilde{\xi}_a(x) \in \psi_m K_{n-1}(y)$. Hence $\tilde{\xi}_a(x) \in K_n(x)$, as required.

We have now shown that (J, γ) is the terminal coalgebra in **Set**—or properly, with notation as in Proposition 5.9, that $U_*(J, \gamma)$ is the terminal coalgebra in **Set**. By Theorem B.1, condition **S** holds, so $U_*(J, \gamma)$ is the universal solution $U_*(I, \iota)$ constructed in §6. Also (I, ι) is the universal solution in **Top**, so there is a unique map $(J, \gamma) \longrightarrow (I, \iota)$ of coalgebras in **Top**. Each component $J(a) \longrightarrow I(a)$ is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism. So (J, γ) is isomorphic to (I, ι) as a coalgebra in **Top**; hence it is the universal solution in **Top**. \square

In many examples the universal solution is especially easy to recognize.

Corollary 9.7 (Crude Recognition Theorem) *Let (\mathbb{A}, M) be an equational system with \mathbb{A} finite. Let $(J, \gamma = \psi^{-1})$ be a fixed point of M in **Top** such that for each $a \in \mathbb{A}$, the space $J(a)$ is nonempty and compact. Suppose further that the spaces $J(a)$ can be metrized in such a way that for each sector $b \xrightarrow{m} a$, the induced map $J(b) \xrightarrow{\psi_m} J(a)$ is a contraction. Then (J, γ) is the universal solution of (\mathbb{A}, M) .*

Proof Since \mathbb{A} and M are finite, there are only finitely many sectors m , so we may choose $\lambda < 1$ such that each map ψ_m is a contraction with constant λ . Since \mathbb{A} is finite and each space $J(a)$ is compact, we may also choose $D \geq 0$ such that $\text{diam}(J(a)) \leq D$ for all $a \in \mathbb{A}$.

We verify condition (b) of the Precise Recognition Theorem. Certainly J is occupied. For the main part of the condition, we have $\text{diam}(V_{m_1, \dots, m_n}^J) \leq \lambda^n D$, and $\inf_{n \in \mathbb{N}} \lambda^n D = 0$. \square

10 Examples

We illustrate the power of the Recognition Theorems by using them to produce examples of universal solutions. We can easily derive Freyd's theorem on the interval, and we give similar characterizations of circles, cubes, simplices and various fractal spaces.

Discrete examples

Even in the relatively trivial case of discrete equational systems, the Recognition Theorems can be useful.

Example 10.1 (Cantor set) Write $\mathbf{1}$ for the terminal category (one object and only the identity arrow). An equational system $(\mathbf{1}, M)$ amounts to a finite set M , and an M -coalgebra is a space X equipped with a map into the M -fold coproduct $M \times X$. The universal solution is the power $M^{\mathbb{N}^+}$ (regarding the set M as a discrete space) together with the isomorphism $\gamma = \psi^{-1} : M^{\mathbb{N}^+} \xrightarrow{\sim} M \times M^{\mathbb{N}^+}$. This can be shown directly, or from the description of the universal solution in §1, or from a Recognition Theorem as follows.

The space $M^{\mathbb{N}^+}$ is compact. It is nonempty if M is, so the coalgebra $(M^{\mathbb{N}^+}, \gamma)$ is occupied. For $m \in M$, the map $\psi_m : M^{\mathbb{N}^+} \longrightarrow M^{\mathbb{N}^+}$ is given by

$$\psi_m(m_1, m_2, \dots) = (m, m_1, m_2, \dots),$$

so condition (c) of the Precise Recognition Theorem holds. Hence $(M^{\mathbb{N}^+}, \gamma)$ is the universal solution. When M has cardinality 2, the universal solution is the standard Cantor set $2^{\mathbb{N}^+}$.

In fact, the homeomorphism type of $M^{\mathbb{N}^+}$ is independent of M for $|M| \geq 2$. This classical fact can be proved as follows. Let $k \geq 2$. Write $k = \{0, \dots, k-1\}$, write $\psi : 2 \times 2^{\mathbb{N}^+} \xrightarrow{\sim} 2^{\mathbb{N}^+}$ for the usual isomorphism, and let $\psi^{(k)} : k \times 2^{\mathbb{N}^+} \xrightarrow{\sim} 2^{\mathbb{N}^+}$ be the composite

$$k \times 2^{\mathbb{N}^+} \xrightarrow[\sim]{\psi + \text{id}} (k-1) \times 2^{\mathbb{N}^+} \xrightarrow[\sim]{\psi + \text{id}} \dots \xrightarrow[\sim]{\psi + \text{id}} 2 \times 2^{\mathbb{N}^+} \xrightarrow[\sim]{\psi} 2^{\mathbb{N}^+}.$$

Then for each $m \in k$, the map $\psi_m^{(k)} : 2^{\mathbb{N}^+} \longrightarrow 2^{\mathbb{N}^+}$ is of the form $\psi_{p_1} \cdots \psi_{p_r}$ for some $r \geq 1$ and $p_1, \dots, p_r \in 2$. Using the metric on $2^{\mathbb{N}^+}$ induced by its embedding into $[0, 1]$ (defined in §1), ψ_0 and ψ_1 are contractions with constant $1/3$; hence each map $\psi_m^{(k)}$ is also a contraction with constant (at most) $1/3$. By the Crude Recognition Theorem, $(2^{\mathbb{N}^+}, (\psi^{(k)})^{-1})$ is the universal solution of $(\mathbf{1}, k)$. In particular, $2^{\mathbb{N}^+} \cong k^{\mathbb{N}^+}$ for all $k \geq 2$.

Example 10.2 (Universal convergent sequence) There is a discrete equational system defined informally by

$$\begin{aligned} X_1 &\cong X_1 \\ X_2 &\cong X_1 + X_2 \end{aligned}$$

(as in the Introduction). Its universal solution is $X_1 = \{0\}$ and $X_2 = \mathbb{N} \cup \{\infty\}$, with X_2 topologized as the Alexandroff one-point compactification of the discrete space \mathbb{N} . This can be shown by an easy application of the Crude Recognition Theorem, metrizing $\mathbb{N} \cup \{\infty\}$ by using the evident homeomorphism with the subspace $\{2^{-n} \mid n \in \mathbb{N}\} \cup \{0\}$ of \mathbb{R} .

A discrete equational system may contain equations of the form $X_i = X_i$, or loops such as $X_1 = X_2, X_2 = X_3, X_3 = X_1$, or infinite chains such as $X_1 = X_2, X_2 = X_3, \dots$. In those cases the universal solution (I, ι) will involve the one-point space, and perhaps other spaces containing isolated points (as in the last example). But if the one-point space is not involved then I is extremely simple:

Proposition 10.3 (Empty or Cantor) *Let (\mathbb{A}, M) be a discrete equational system with universal solution (I, ι) . Suppose that $|I(a)| \neq 1$ for all $a \in \mathbb{A}$. Then each space $I(a)$ is either empty or the Cantor set.*

This is closely related to the classical fact that, up to homeomorphism, the empty set and the Cantor set are the only totally disconnected compact metrizable spaces with no isolated points [HY]. In fact, our proposition together with the discrete realizability theorem (C.7) leads to a new proof of this fact [Lei2]; see also Appendix C.

Proof Define $J : \mathbb{A} \longrightarrow \mathbf{Top}$ by

$$J(a) = \begin{cases} \emptyset & \text{if } I(a) = \emptyset \\ 2^{\mathbb{N}^+} & \text{if } I(a) \neq \emptyset. \end{cases}$$

For each $a \in \mathbb{A}$, let $k(a) = \left| \sum_{b: I(b) \neq \emptyset} M(b, a) \right| \in \mathbb{N}$ and choose an isomorphism between the sets $\sum_{b: I(b) \neq \emptyset} M(b, a)$ and $k(a)$. (We continue to write n for the n -element set $\{0, \dots, n-1\}$.) Then for all $a \in \mathbb{A}$,

$$(M \otimes J)(a) = \sum_{b \in \mathbb{A}} M(b, a) \times J(b) \cong \sum_{b: I(b) \neq \emptyset} M(b, a) \times 2^{\mathbb{N}^+} \cong k(a) \times 2^{\mathbb{N}^+}.$$

Define an isomorphism $\gamma_a : J(a) \xrightarrow{\sim} (M \otimes J)(a)$ for each $a \in \mathbb{A}$ as follows. If $I(a) = \emptyset$ then $J(a) = \emptyset = (M \otimes J)(a)$, and we put $\gamma_a = 1_\emptyset$. If $I(a) \neq \emptyset$ then $J(a) = 2^{\mathbb{N}^+}$ and we put

$$\gamma_a = \left(2^{\mathbb{N}^+} \xrightarrow[\sim]{(\psi^{(k(a))})^{-1}} k(a) \times 2^{\mathbb{N}^+} \cong (M \otimes J)(a) \right)$$

where for $k \geq 2$, the homeomorphism

$$\psi^{(k)} : k \times 2^{\mathbb{N}^+} \xrightarrow{\sim} 2^{\mathbb{N}^+}$$

is defined as in Example 10.1, and $\psi^{(1)} = \text{id}$. We show that this fixed point (J, γ) satisfies condition (c) of the Precise Recognition Theorem.

Certainly J is occupied, and each space $J(a)$ is compact and can be equipped with the usual metric. Now take a complex (a_\bullet, m_\bullet) . Write $\psi = \gamma^{-1}$. For each $r \in \mathbb{N}$, either $k(a_r) = 1$, in which case $\psi_{m_{r+1}}$ is an isometry, or $k(a_r) \geq 2$, in which case $\psi_{m_{r+1}}$ is a contraction with constant $1/3$. The latter case arises infinitely often: for if not, there is some $s \in \mathbb{N}$ for which $1 = k(a_s) = k(a_{s+1}) = \dots$, and then $|I(a_s)| = 1$, contrary to hypothesis. Moreover, $\text{diam}(J(a_r)) \leq 1$ for each r . So

$$\text{diam}(V_{m_1, \dots, m_r}^J) = \text{diam}(\psi_{m_1} \dots \psi_{m_r} J(a_r)) \rightarrow 0 \text{ as } r \rightarrow \infty$$

and therefore condition (c) of the Precise Recognition Theorem is satisfied, as required. \square

Example 10.4 (Walks) Consider again the example from §1 of spaces of walks, but suppose now that we change the rule at 0 to read ‘if at position 0, step right’. Thus, the first equation of the system changes from ‘ $W_0 = W_0$ ’ to ‘ $W_0 = W_1$ ’. Each of the spaces W_n making up the universal solution is now infinite, and in particular $|W_n| > 1$ for all $n \in \mathbb{N}$. So by Proposition 10.3, W_n is homeomorphic to the Cantor set for all $n \in \mathbb{N}$.

Contrast the universal solution (W_n) of the original set of rules; there, $|W_0| = 1$, and since it is possible to walk to 0 from any position n , each of the spaces W_n has at least one isolated point.

Non-discrete examples

Example 10.5 (Interval) We finally prove the topological Freyd theorem (2.2). So far we have verified that the (\mathbb{A}, M) concerned is an equational system, and exhibited an M -coalgebra (J, γ) (previously written (I, ι)) with $J(0) = \{\star\}$ and $J(1) = [0, 1]$. We apply the Crude Recognition Theorem (9.7). Both spaces $J(a)$ are nonempty, compact, and can be metrized in the usual way. Evidently γ is invertible, so we have a fixed point $(J, \gamma = \psi^{-1})$. For a sector $m : b \dashrightarrow a$ in (\mathbb{A}, M) , the induced map $\psi_m : J(b) \longrightarrow J(a)$ is either constant or, in the case that m is one of two sectors $1 \dashrightarrow 1$, it is one of the two maps

$$\begin{array}{ccc} [0, 1] & \longrightarrow & [0, 1] \\ t & \longmapsto & t/2 \\ t & \longmapsto & (t+1)/2. \end{array}$$

All of the maps ψ_m are therefore contractions. Hence (J, γ) is the universal solution.

Freyd’s Theorem expresses $[0, 1]$ as two copies of itself glued end to end. Two can be replaced by any larger number. Thus, for each $k \geq 2$ there is a corresponding equational system $(\mathbb{A}, M^{(k)})$, with \mathbb{A} as above and, for instance, $|M^{(k)}(1, 1)| = k$. The multiplication map $k \cdot - : [0, 1] \longrightarrow [0, k]$ puts an $M^{(k)}$ -coalgebra structure $\gamma^{(k)}$ on the functor J , and the same argument shows that $(J, \gamma^{(k)})$ is the universal solution. So the interval, like the Cantor set (Example 10.1), is recursively realizable in infinitely many ways.

Example 10.6 (Circle) The recursive description of the interval can easily be extended to give a recursive description of the circle S^1 . The circle is the

coequalizer of the diagram

$$\{\star\} \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} [0, 1],$$

and the crucial observation is that all of these spaces and maps appear in the universal solution of the Freyd system.

Let \mathbb{A} be the following category with 3 objects and 2 non-identity arrows:

$$0 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 1 \qquad 2.$$

The idea is to extend the Freyd system to an equational system on \mathbb{A} , in such a way that for any $X \in \langle \mathbb{A}, \mathbf{Top} \rangle$, the space $(M \otimes X)(2)$ is the coequalizer of $X\sigma, X\tau : X(0) \rightrightarrows X(1)$. So, define a module $M : \mathbb{A} \rightarrow \mathbf{Top}$ as follows. The restriction of M to the full subcategory $\{0, 1\}$ of \mathbb{A} is the module of the Freyd system. For all $a \in \{0, 1, 2\}$, $M(2, a) = \emptyset$. Finally, $|M(0, 2)| = |M(1, 2)| = 1$.

There is a nondegenerate functor $J : \mathbb{A} \rightarrow \mathbf{Top}$ given by

$$\{\star\} \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} [0, 1] \qquad S^1.$$

The functor $M \otimes J$ can naturally be identified with

$$\{\star\} \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{2} \end{array} [0, 2] \qquad [0, 1]/(0 = 1)$$

(the rightmost object being $[0, 1]$ with its endpoints identified), so there is an evident isomorphism $\gamma : J \xrightarrow{\sim} M \otimes J$.

We show that (J, γ) is the universal solution of (\mathbb{A}, M) using the Crude Recognition Theorem. Each of the spaces $J(a)$ is compact and nonempty. Write $\psi = \gamma^{-1}$. We have to check that the spaces $J(a)$ can be metrized in such a way that for each sector $m : b \rightarrow a$, the map $\psi_m : J(b) \rightarrow J(a)$ is a contraction. For the sectors m in the Freyd system, we have already shown this in Example 10.5. For the sector $0 \rightarrow 2$, it is trivial. The only remaining sector is $1 \rightarrow 2$, whose induced map is the quotient map $[0, 1] \rightarrow S^1$, and this is a contraction if a suitably scaled-down metric on S^1 is chosen. So the Crude Recognition Theorem applies, as claimed.

Products

Given recursive realizations of spaces S and S' , there arises, in a canonical way, a recursive realization of the product space $S \times S'$. This follows from Proposition 10.8 below. We use the fact that the category of equational systems has finite products (§2).

Lemma 10.7 *Let \mathbb{B} and \mathbb{B}' be small categories, let $Y : \mathbb{B}^{\text{op}} \rightarrow \mathbf{Set}$ and $Y' : \mathbb{B}'^{\text{op}} \rightarrow \mathbf{Set}$ be functors, and let $X : \mathbb{B} \rightarrow \mathbf{Top}$ and $X' : \mathbb{B}' \rightarrow \mathbf{Top}$ be functors taking values in compact Hausdorff spaces. Then*

$$(Y \times Y') \otimes (X \times X') \cong (Y \otimes X) \times (Y' \otimes X')$$

where on the left-hand side, ‘ \times ’ is used in the sense of Lemma 2.12.

Proof We use the fact that if K is a compact Hausdorff space then $K \times - : \mathbf{Top} \longrightarrow \mathbf{Top}$ preserves colimits. We also use a formula from Appendix A:

$$Y \otimes X = \lim_{\rightarrow (b,y) \in \mathbb{E}(Y)} X(b).$$

Now

$$\begin{aligned} (Y \times Y') \otimes (X \times X') &\cong \lim_{\rightarrow ((b,b'),(y,y')) \in \mathbb{E}(Y \times Y')} (X \times X')(b, b') \\ &\cong \lim_{\rightarrow (b,y) \in \mathbb{E}(Y)} \lim_{\rightarrow (b',y') \in \mathbb{E}(Y')} X(b) \times X'(b') \\ &\cong \lim_{\rightarrow (b,y) \in \mathbb{E}(Y)} X(b) \times \lim_{\rightarrow (b',y') \in \mathbb{E}(Y')} X'(b') \\ &\cong (Y \otimes X) \times (Y' \otimes X'). \end{aligned}$$

□

Proposition 10.8 (Universal solution of product) *Let (\mathbb{A}, M) and (\mathbb{A}', M') be equational systems with universal solutions (I, ι) and (I', ι') , respectively, in \mathbf{Top} . Then the product $(\mathbb{A}, M) \times (\mathbb{A}', M') = (\mathbb{A} \times \mathbb{A}', M \times M')$ in the category of equational systems has universal solution $(I \times I', \iota \times \iota')$ in \mathbf{Top} .*

Proof The functor $I \times I' : \mathbb{A} \times \mathbb{A}' \longrightarrow \mathbf{Top}$ is nondegenerate: for

$$U \circ (I \times I') \cong (U \circ I) \times (U \circ I') : \mathbb{A} \times \mathbb{A}' \longrightarrow \mathbf{Set}$$

is nondegenerate by Lemma 2.12(c), and $I(a) \times I'(a')$ is compact Hausdorff for all $a \in \mathbb{A}$, $a' \in \mathbb{A}'$. By Lemma 10.7, we have a natural isomorphism

$$\iota \times \iota' : I \times I' \xrightarrow{\sim} (M \otimes I) \times (M' \otimes I') \cong (M \times M') \otimes (I \times I').$$

Also, $I \times I'$ is occupied since I and I' are. To finish the proof it remains only to verify that $(I \times I', \iota \times \iota')$ satisfies the main condition in (c) of the Precise Recognition Theorem, and this follows from the fact that it is satisfied by (I, ι) and (I', ι') . □

Example 10.9 (Cubes) Let (\mathbb{A}, M) be the Freyd system. Then by Proposition 10.8, (\mathbb{A}^2, M^2) has a universal solution (I, ι) satisfying $I(1, 1) = [0, 1]^2$. Informally, the self-similarity equations are

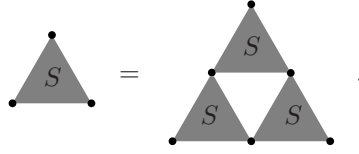
The diagram shows four equations representing self-similarity in the Freyd system. Each equation consists of two parts separated by an equals sign. The first part is a simple geometric object, and the second part is a more complex version of the same object, illustrating how the object is built from smaller copies of itself. 1. A single point equals a single point. 2. A line segment equals a line segment. 3. A vertical line segment equals a vertical line segment with a point in the middle. 4. A square equals a 2x2 grid of squares.

A similar statement holds in higher dimensions.

Further non-discrete examples

An **iterated function system** on \mathbb{R}^d is a family ψ_0, \dots, ψ_n ($n \geq 0$) of contractions $\mathbb{R}^d \longrightarrow \mathbb{R}^d$. By a theorem of Hutchinson [Hut], there is a unique nonempty compact subset S of \mathbb{R}^d satisfying $S = \bigcup_{i=0}^n \psi_i S$, the **attractor** of the system. Various familiar self-similar spaces arise in this way.

Example 10.10 (Sierpiński simplices) Let $n \in \mathbb{N}$ and let s_0, \dots, s_n be affinely independent points of \mathbb{R}^n . For each $i \in \{0, \dots, n\}$, write $\psi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for the scaling with scale factor $1/2$ and fixed point s_i . The **Sierpiński simplex** with vertices s_0, \dots, s_n is the attractor of the iterated function system (ψ_0, \dots, ψ_n) . When $n = 1$, it is the closed interval with endpoints s_0 and s_1 . When $n = 2$, it is the usual Sierpiński triangle or gasket S , which satisfies an isomorphism expressed informally as



Now take any $n \in \mathbb{N}$ and s_0, \dots, s_n as above, and write S for the resulting Sierpiński simplex. We construct an equational system whose universal solution is S (equipped with some extra structure).

Let \mathbb{A} be the category with objects 0 and 1 and non-identity arrows $\sigma_0, \dots, \sigma_n : 0 \longrightarrow 1$. Define an equivalence relation \sim on $\{0, \dots, n\}^2$ by $(i, j) \sim (i', j')$ if and only if $\{i, j\} = \{i', j'\}$, and write $[i, j]$ for the equivalence class of (i, j) . Define $M : \mathbb{A} \dashrightarrow \mathbb{A}$ by

$$\begin{array}{ccc}
 & & \begin{array}{ccc} M(-, 0) & \xrightarrow{\sigma_0 \cdot -} & M(-, 1) \\ & \vdots & \\ & \xrightarrow{\sigma_n \cdot -} & \end{array} \\
 & & \hline
 & & \begin{array}{ccc} & \xrightarrow{[0, 0]} & \\ & \vdots & \\ & \xrightarrow{[n, n]} & \end{array} \\
 \begin{array}{ccc} M(0, -) & & \{0, \dots, n\}^2 / \sim \\ \uparrow \sigma_0 \quad \vdots \quad \uparrow \sigma_n & & \\ M(1, -) & & \end{array} & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \{id\} & & \\ \vdots & & \\ \vdots & & \end{array} & \begin{array}{ccc} \vdots & & \\ \vdots & & \\ \vdots & & \end{array} & \begin{array}{ccc} \vdots & & \\ \vdots & & \\ \vdots & & \end{array} \\ \vdots & & \vdots \\ \emptyset & \longrightarrow & \{0, \dots, n\} \end{array} & \begin{array}{ccc} \vdots & & \\ \vdots & & \\ \vdots & & \end{array} & \begin{array}{ccc} \vdots & & \\ \vdots & & \\ \vdots & & \end{array} \\
 \end{array}$$

Then (\mathbb{A}, M) is an equational system.

Any space X_1 equipped with distinct basepoints x_0, \dots, x_n determines a nondegenerate functor $X : \mathbb{A} \longrightarrow \mathbf{Top}$, with $X(0) = \{\star\}$ and $X(1) = X_1$. Then $M \otimes X$ is the functor determined by the quotient space

$$\frac{\{0, \dots, n\} \times X_1}{(i, x_j) = (j, x_i) \text{ for all } i, j}$$

with basepoints $(0, x_0), \dots, (n, x_n)$.

In particular, (S, s_0, \dots, s_n) determines a nondegenerate functor $J : \mathbb{A} \longrightarrow \mathbf{Top}$. The function

$$\begin{array}{ccc} \{0, \dots, n\} \times S & \longrightarrow & S \\ (i, s) & \longmapsto & \psi_i(s) \end{array} \quad (44)$$

induces a map $(M \otimes J)(1) \longrightarrow S = J(1)$, since $\psi_i(s_j) = \frac{1}{2}(s_i + s_j) = \psi_j(s_i)$ for all i, j . This map is surjective by definition of S . It is injective since each map ψ_i is injective and

$$\psi_i S \cap \psi_j S = \{\psi_i(s_j)\} = \{\psi_j(s_i)\}$$

whenever $i \neq j$. It also preserves basepoints. So we have an isomorphism $\psi : M \otimes J \longrightarrow J$. The spaces $J(0) = \{\star\}$ and $J(1) = S$ are nonempty and compact, and the structure maps $\psi_i : J(1) \longrightarrow J(1)$ are contractions, so by the Crude Recognition Theorem, (J, ψ^{-1}) is the universal solution of (\mathbb{A}, M) .

We have therefore realized the n -dimensional Sierpiński simplex as the solution of an equational system, in a way that formalizes the idea that it is homeomorphic to a gluing of $(n+1)$ half-sized copies of itself.

Example 10.11 (Iterated function systems) More generally, let (ψ_0, \dots, ψ_n) be an iterated function system on \mathbb{R}^d (some $d \in \mathbb{N}$). Write S for its attractor, and s_i for the fixed point of ψ_i . Suppose that ψ_0, \dots, ψ_n are injective, that s_0, \dots, s_n are distinct, and that if $\psi_i(s) = \psi_j(t)$ with $i \neq j$ and $s, t \in S$ then $s, t \in \{s_0, \dots, s_n\}$. Then the space S can be realized by a finite equational system, as follows.

Define an equivalence relation \sim on $\{0, \dots, n\}^2$ by $(i, j) \sim (i', j') \iff \psi_i(s_j) = \psi_{i'}(s_{j'})$, and write $[i, j]$ for the equivalence class of (i, j) . Proceeding exactly as in the previous example, this equivalence relation gives rise to an equational system (\mathbb{A}, M) ; the space S with basepoints s_0, \dots, s_n determines a nondegenerate functor $J : \mathbb{A} \longrightarrow \mathbf{Top}$; the maps ψ_i determine an isomorphism $M \otimes J \xrightarrow{\sim} J$; and by the Crude Recognition Theorem, this is the universal solution of (\mathbb{A}, M) .

Even for iterated function systems not within the scope of this example, the attractor may still have a straightforward description as a universal solution: $[0, 1]^n$ in Example 10.9, for instance.

Example 10.12 (Barycentric subdivision) Barycentric subdivision expresses the n -simplex Δ^n as $(n+1)!$ smaller copies of itself glued together along simplices of lower dimension. This self-similarity can be formalized as follows.

Let Δ_{inj} be the category whose objects are the nonempty finite totally ordered sets $[n] = \{0, \dots, n\}$ ($n \in \mathbb{N}$) and whose maps are the order-preserving injections. For each $n, m \in \mathbb{N}$, put

$$M([n], [m]) = \{\text{chains } \emptyset \subset Q(0) \subset \dots \subset Q(n) \subseteq [m]\}$$

where \subset means proper subset. (This can be regarded as the set of n -simplices occurring in the barycentric subdivision of Δ^m . It is empty unless $n \leq m$.) The idea can be seen in Figure 10.4: the 1-simplex in bold and the shaded 2-simplex correspond respectively to

$$\begin{aligned} (\emptyset \subset \{0, 2\} \subset \{0, 1, 2\}) &\in M([1], [2]), \\ (\emptyset \subset \{0\} \subset \{0, 2\} \subset \{0, 1, 2\}) &\in M([2], [2]). \end{aligned}$$

An element of $M([n], [m])$ can be regarded as an order-preserving injection $[n] \longrightarrow \mathcal{P}_{\neq \emptyset}[m]$, where $\mathcal{P}_{\neq \emptyset}$ denotes the set of nonempty subsets ordered by inclusion. By using direct images, $\mathcal{P}_{\neq \emptyset}[m]$ is functorial in $[m]$, so M defines a

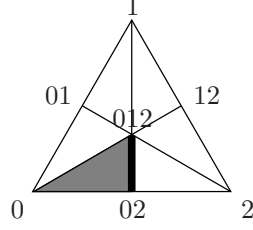


Figure 10.4: Barycentric subdivision of Δ^2

module $\Delta_{\text{inj}} \dashrightarrow \Delta_{\text{inj}}$. It can be checked that M is nondegenerate using the explicit conditions **ND1** and **ND2** (§4). And clearly M is finite, so (Δ_{inj}, M) is an equational system.

We will show that the universal solution is given by the standard topological simplex functor $\Delta^\bullet : \Delta_{\text{inj}} \longrightarrow \mathbf{Top}$. For each $n \in \mathbb{N}$, fix an affinely independent sequence e_0^n, \dots, e_n^n of points in \mathbb{R}^n , and let Δ^n be their convex hull. Then for each map $f : [n] \longrightarrow [m]$ in Δ_{inj} there is a unique affine map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ sending e_j^n to $e_{f(j)}^m$ for each j , which restricts to a map $\Delta f = f_* : \Delta^n \longrightarrow \Delta^m$. It is straightforward to check that $U \circ \Delta^\bullet : \Delta_{\text{inj}} \longrightarrow \mathbf{Set}$ is nondegenerate, again using conditions **ND1** and **ND2**. (Roughly speaking, this expresses the fact that the intersection of two faces of a simplex, if not empty, is again a face.) Moreover, each space Δ^n is compact Hausdorff, so Δ^\bullet is nondegenerate.

We construct an isomorphism $M \otimes \Delta^\bullet \cong \Delta^\bullet$. (This expresses the fact that we really do have a subdivision.) By the universal property of tensor product (Appendix A), a natural transformation $\psi : M \otimes \Delta^\bullet \longrightarrow \Delta^\bullet$ amounts to a choice, for each sector $Q : [n] \dashrightarrow [m]$, of a map $\psi_Q : \Delta^n \longrightarrow \Delta^m$, satisfying the naturality condition $\psi_{fQg} = f_* \circ \psi_Q \circ g_*$ for all f , Q and g . Indeed, given such a Q , there is a unique affine map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$e_j^n \mapsto \frac{1}{|Q(j)|} \sum_{i \in Q(j)} e_i^m$$

for all $j \in [n]$, and this restricts to a map $\psi_Q : \Delta^n \longrightarrow \Delta^m$. The naturality condition is easily verified.

This natural transformation $\psi : M \otimes \Delta^\bullet \longrightarrow \Delta^\bullet$ is indeed an isomorphism. To prove this, it suffices to show that for each $m \in \mathbb{N}$, the continuous map

$$\psi : M(-, [m]) \otimes \Delta^\bullet \longrightarrow \Delta^m \quad (45)$$

is a homeomorphism. Its domain is compact and its codomain Hausdorff, so in fact it suffices to show that it is a bijection. The inverse is constructed as follows. Let $s \in \Delta^m$; then $s = \sum_{i=0}^m s_i e_i^m$ with $s_i \geq 0$ and $\sum s_i = 1$. There are unique $n \in \mathbb{N}$ and $s'_0 > \dots > s'_n > s'_{n+1} = 0$ such that

$$\{s'_0, \dots, s'_n, s'_{n+1}\} = \{s_0, \dots, s_m, 0\},$$

and we may define $q : [m] \longrightarrow [n+1]$ by $s_i = s'_{q(i)}$. For $j \in [n]$, put

$$Q(j) = q^{-1}\{0, \dots, j\}, \quad t_j = (s'_j - s'_{j+1})|Q(j)|, \quad t = \sum_{j=0}^n t_j e_j^n.$$

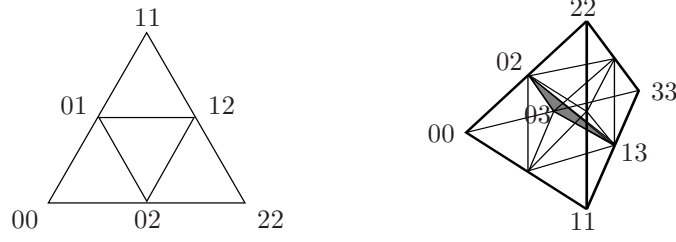


Figure 10.5: Edgewise subdivisions of Δ^2 and Δ^3

A series of straightforward checks shows that $Q \in M([n], [m])$, $t \in \Delta^n$, and the inverse to (45) is given by $s \mapsto Q \otimes t$.

We now verify condition (b) of the Precise Recognition Theorem. A standard calculation [Hat, 2.21] shows that for any $Q : [n] \dashrightarrow [m]$,

$$\text{diam}(\psi_Q \Delta^n) \leq \frac{m}{m+1} \text{diam}(\Delta^m)$$

in the Euclidean metric. More generally, if

$$[n_r] \xrightarrow{Q_r} \cdots \xrightarrow{Q_1} [n_0]$$

then the same method shows that

$$\begin{aligned} \text{diam}(V_{Q_1, \dots, Q_r}^J) &= \text{diam}(\psi_{Q_1} \cdots \psi_{Q_r} \Delta^{n_r}) \\ &\leq \left(\frac{n_{r-1}}{n_{r-1} + 1} \right) \cdots \left(\frac{n_0}{n_0 + 1} \right) \text{diam}(\Delta^{n_0}) \\ &\leq \left(\frac{n_0}{n_0 + 1} \right)^r \text{diam}(\Delta^{n_0}). \end{aligned}$$

Condition (b) follows.

Hence the topological simplex functor Δ^\bullet is the universal solution to the equational system embodying the combinatorial process of barycentric subdivision.

Examples 10.1 (Cantor set) and 10.5 (interval) showed that the same space may have multiple different recursive realizations. It may be thought that the different realizations in those examples were not dramatically different. But the previous example and the next illustrate a greater contrast.

Example 10.13 (Edgewise subdivision) The topological simplex functor $\Delta^\bullet : \Delta_{\text{inj}} \longrightarrow \mathbf{Top}$ can also be characterized by edgewise subdivision. This subdivision (Figure 10.5 and [Freu]) expresses Δ^n as 2^n smaller copies of itself glued together. It can be viewed as an equational system (Δ_{inj}, M) where

$$\begin{aligned} M([n], [m]) &= \{ \text{order-preserving injections } (p, q) : [n] \longrightarrow [m] \times [m] \\ &\quad \text{such that } p(n) \leq q(0) \} \end{aligned}$$

and $[m] \times [m]$ is the product in the category of posets. (Again, this set indexes the n -simplices occurring in the subdivision of Δ^m , and again, it is empty unless

$n \leq m$.) For instance, the shaded 2-simplex inside the 3-simplex in Figure 10.5 is the sector $[2] \dashrightarrow [3]$ given by the order-preserving injection $[2] \longrightarrow [3] \times [3]$ with image $\{(0, 2), (0, 3), (1, 3)\}$. Again it can be shown that $\Delta^\bullet : \Delta_{\text{inj}} \longrightarrow \mathbf{Top}$, with a canonical M -coalgebra structure, is the universal solution.

A Appendix: Modules

Here we state some basic features of the theory of modules over categories, continuing the remarks at the end of the Introduction.

Much of this theory can be understood by analogy with the theory of modules and bimodules in the ordinary sense of algebra. It was already noted in the Introduction that when \mathbb{A} and \mathbb{B} are monoids, seen as one-object categories, a module $\mathbb{B} \dashrightarrow \mathbb{A}$ is a set with compatible left \mathbb{A} - and right \mathbb{B} -actions. If we work with categories enriched in abelian groups, then a one-object category is exactly a ring and a module $\mathbb{B} \dashrightarrow \mathbb{A}$ between rings \mathbb{A} and \mathbb{B} is exactly an (\mathbb{A}, \mathbb{B}) -bimodule. In fact, the theory of categorical modules can be developed in the generality of enriched categories, and this general theory contains many parts of the theory of algebraic (bi)modules. For example, there are notions of tensor product and flatness of categorical modules, generalizing the notions from algebra.

Indeed, given rings A , B and C , an (A, B) -bimodule M , and a (B, C) -bimodule N , there arises an (A, C) -bimodule $M \otimes_B N$. There is a similar tensor product of categorical modules:

$$\mathbb{C} \xrightarrow{N} \mathbb{B} \dashrightarrow \mathbb{A} \quad \text{gives rise to} \quad \mathbb{C} \xrightarrow{M \otimes N} \mathbb{A}.$$

Here $M \otimes N$ is defined by the coend formula

$$(M \otimes N)(c, a) = \int^b M(b, a) \times N(c, b).$$

Coends are explained in [Mac, Ch. IX]; concretely,

$$(M \otimes N)(c, a) = \left(\sum_{b \in \mathbb{B}} M(b, a) \times N(c, b) \right) / \sim$$

where \sim is the equivalence relation generated by $(mg, n) \sim (m, gn)$ for all $m \in M(b, a)$, $g \in \mathbb{B}(b', b)$ and $n \in N(c, b')$. The element of $(M \otimes N)(c, a)$ represented by $(m, n) \in M(b, a) \times N(c, b)$ is written $m \otimes n$. The tensor product of modules is associative and unital up to coherent isomorphism. (More precisely, categories, modules, and their maps form a bicategory: [Bor, 7.8.2].)

In the special case where \mathbb{C} is the terminal category $\mathbf{1}$, the tensor product construction gives for each module $M : \mathbb{B} \dashrightarrow \mathbb{A}$ and functor $X : \mathbb{B} \longrightarrow \mathbf{Set}$ a new functor $M \otimes X : \mathbb{A} \longrightarrow \mathbf{Set}$. Concretely,

$$(M \otimes X)(a) = \int^b M(b, a) \times X(b) = \left(\sum_{b \in \mathbb{B}} M(b, a) \times X(b) \right) / \sim$$

where $a \in \mathbb{A}$ and \sim is as above. An equivalent formulation uses the notion of category of elements (defined after Example 2.5):

$$(M \otimes X)(a) = \lim_{\rightarrow (b, m) \in \mathbb{E}(M(-, a))} X(b),$$

where the right-hand side is a colimit over objects (b, m) of $\mathbb{E}(M(-, a))$, the category of elements of $M(-, a) : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set}$. Both the coend and colimit formulations continue to make sense when X takes values not in \mathbf{Set} but in some other category \mathcal{E} with small colimits; thus,

$$X : \mathbb{B} \longrightarrow \mathcal{E} \quad \text{and} \quad M : \mathbb{B}^{\text{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$$

give rise to $M \otimes X : \mathbb{A} \longrightarrow \mathcal{E}$. This product $M \otimes X$ can be characterized by a universal property: for any functor $Z : \mathbb{A} \longrightarrow \mathcal{E}$, the natural transformations $\psi : M \otimes X \longrightarrow Z$ are in natural bijection with the families

$$\left(X(b) \xrightarrow{\psi_m} Z(a) \right)_{b \dashrightarrow^m a}$$

of maps in \mathcal{E} (indexed over all $b \in \mathbb{B}$, $a \in \mathbb{A}$ and $m \in M(b, a)$) such that $\psi_{fmg} = (Zf) \circ \psi_m \circ (Xg)$ for all

$$b' \xrightarrow{g} b \dashrightarrow^m a \xrightarrow{f} a'.$$

In the even more special case $\mathbb{C} = \mathbb{A} = \mathbf{1}$, the tensor product construction gives for each pair of functors $X : \mathbb{B} \longrightarrow \mathbf{Set}$, $Y : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set}$ a set

$$Y \otimes X = \int^b Y(b) \times X(b) = \left(\sum_{b \in \mathbb{B}} Y(b) \times X(b) \right) / \sim = \lim_{\rightarrow (b, y) \in \mathbb{E}(Y)} X(b).$$

(Again, the construction also makes sense for $X : \mathbb{B} \longrightarrow \mathcal{E}$ and $Y : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set}$, for suitable categories \mathcal{E} ; then $Y \otimes X \in \mathcal{E}$.) In fact, the general construction can be written in terms of this very special case: for modules M and N as above, and $a \in \mathbb{A}$, $c \in \mathbb{C}$, we have functors

$$N(c, -) : \mathbb{B} \longrightarrow \mathbf{Set}, \quad M(-, a) : \mathbb{B}^{\text{op}} \longrightarrow \mathbf{Set},$$

and then

$$(M \otimes N)(c, a) = M(-, a) \otimes N(c, -).$$

The notion of commutative diagram in a category \mathbb{A} can be extended to include elements of a module $M : \mathbb{A} \dashrightarrow \mathbb{A}$. For instance, the diagram

$$\begin{array}{ccccc} a_2 & \dashrightarrow^{m_2} & a_1 & \dashrightarrow^{m_1} & a_0 \\ f_2 \downarrow & & \downarrow f_1 & & \downarrow f_0 \\ a'_2 & \dashrightarrow^{m'_2} & a'_1 & \dashrightarrow^{m'_1} & a'_0 \end{array}$$

is said to **commute** if $m'_2 f_2 = f_1 m_2$ and $m'_1 f_1 = f_0 m_1$. We never attempt to compose paths containing more than one crossed arrow \dashrightarrow .

B Appendix: Solvability

Here we finish the proof of:

Theorem B.1 (Existence of universal solution) *Let (\mathbb{A}, M) be an equational system. The following are equivalent:*

- a. (\mathbb{A}, M) satisfies the solvability condition **S** of §6
- b. (\mathbb{A}, M) has a universal solution in **Top**
- c. (\mathbb{A}, M) has a universal solution in **Set**.

*In that case, the universal solution in **Set** is the underlying coalgebra in **Set** of the universal solution in **Top**.*

We proved (a) \implies (b) in §8. We proved (b) \implies (c), and the final sentence, as Proposition 5.9. It remains to prove (c) \implies (a).

Fix an equational system (\mathbb{A}, M) . In this appendix, ‘ M -coalgebra’ means ‘ M -coalgebra in **Set**’. We constructed a functor $I : \mathbb{A} \longrightarrow \mathbf{Set}$ in §6; it is defined regardless of whether **S** holds.

Lemma B.2 *The following conditions on (\mathbb{A}, M) are equivalent:*

- a. (\mathbb{A}, M) satisfies **S**
- b. the functor $I : \mathbb{A} \longrightarrow \mathbf{Set}$ is nondegenerate
- c. there exist a nondegenerate functor $J : \mathbb{A} \longrightarrow \mathbf{Set}$ and a natural transformation $I \longrightarrow J$.

Proof We proved (a) \implies (b) as Proposition 7.9, and (b) \implies (c) is trivial. For (c) \implies (a), let γ be a natural transformation from I to a nondegenerate functor $J : \mathbb{A} \longrightarrow \mathbf{Set}$. By definition, $I = \Pi_0 \circ J$, so γ corresponds under the adjunction

$$\mathbf{Cat} \begin{array}{c} \xrightarrow{\Pi_0} \\ \xleftarrow{D} \end{array} \mathbf{Set}$$

to a natural transformation $\bar{\gamma} : J \longrightarrow D \circ J$. This in turn corresponds to a functor $F : \mathbb{E}(J) \longrightarrow \mathbb{E}(J)$ making the following triangle commute:

$$\begin{array}{ccc} \mathbb{E}(J) & \xrightarrow{F} & \mathbb{E}(J) \\ & \searrow \text{pr} & \swarrow \text{pr} \\ & \mathbb{A} & \end{array}$$

where pr denotes a projection. Now, condition **S** says that if \mathbb{K} is either of the categories $(\bullet \longrightarrow \bullet \longleftarrow \bullet)$ or $(\bullet \rightrightarrows \bullet)$, then for any functor $G : \mathbb{K} \longrightarrow \mathbb{E}(J)$, the composite $\text{pr} \circ G$ admits a cone. But given such a G , nondegeneracy of J implies that $F \circ G$ admits a cone, so $\text{pr} \circ F \circ G = \text{pr} \circ G$ admits a cone, as required. \square

To prove (c) \implies (b) of Theorem B.1, we will have to exploit the existence of a terminal object in the category of M -coalgebras; hence we will need a good supply of objects of that category.

For each complex (a_\bullet, m_\bullet) , we construct a representable-type coalgebra. Its underlying functor is

$$H^{(a_\bullet, m_\bullet)} = \sum_{n \in \mathbb{N}} \mathbb{A}(a_n, -) : \mathbb{A} \longrightarrow \mathbf{Set}.$$

Any representable functor is flat, so $H^{(a_\bullet, m_\bullet)}$ is nondegenerate by Theorem 4.11. Also

$$(M \otimes H^{(a_\bullet, m_\bullet)})(b) \cong \sum_{n \in \mathbb{N}} (M \otimes \mathbb{A}(a_n, -))(b) \cong \sum_{n \in \mathbb{N}} M(a_n, b),$$

so an M -coalgebra structure on $H^{(a_\bullet, m_\bullet)}$ amounts to a natural transformation

$$\sum_{n \in \mathbb{N}} \mathbb{A}(a_n, -) \longrightarrow \sum_{n \in \mathbb{N}} M(a_n, -).$$

There is a unique such transformation sending 1_{a_n} to $m_{n+1} \in M(a_{n+1}, a_n)$ for each $n \in \mathbb{N}$; let $\theta^{(a_\bullet, m_\bullet)}$ be the corresponding coalgebra structure on $H^{(a_\bullet, m_\bullet)}$.

This defines an M -coalgebra $(H^{(a_\bullet, m_\bullet)}, \theta^{(a_\bullet, m_\bullet)})$ for each object (a_\bullet, m_\bullet) of $\mathbb{E}(\mathcal{J})$. Moreover, any map $f_\bullet : (a_\bullet, m_\bullet) \longrightarrow (a'_\bullet, m'_\bullet)$ in $\mathbb{E}(\mathcal{J})$ induces a map

$$H^{(a'_\bullet, m'_\bullet)} = \sum_{n \in \mathbb{N}} \mathbb{A}(a'_n, -) \xrightarrow{\sum f_n^*} \sum_{n \in \mathbb{N}} \mathbb{A}(a_n, -) = H^{(a_\bullet, m_\bullet)}$$

respecting the coalgebra structures. So we have a functor

$$(H^\bullet, \theta^\bullet) : \mathbb{E}(\mathcal{J})^{\text{op}} \longrightarrow \mathbf{Coalg}(M, \mathbf{Set}).$$

Having defined the representable-type coalgebras, we prove a Yoneda-type lemma.

Let (X, ξ) be an M -coalgebra. For each complex (a_\bullet, m_\bullet) , write $(X, \xi)(a_\bullet, m_\bullet)$ for the set of resolutions along (a_\bullet, m_\bullet) in (X, ξ) , that is, sequences $(x_n \in X(a_n))_{n \in \mathbb{N}}$ such that $\xi_{a_n}(x_n) = m_{n+1} \otimes x_{n+1}$ for all n . This defines a functor $(X, \xi) : \mathbb{E}(\mathcal{J}) \longrightarrow \mathbf{Set}$.

Lemma B.3 ('Yoneda') *There is a bijection*

$$\mathbf{Coalg}(M, \mathbf{Set}) \left((H^{(a_\bullet, m_\bullet)}, \theta^{(a_\bullet, m_\bullet)}), (X, \xi) \right) \cong (X, \xi)(a_\bullet, m_\bullet)$$

natural in $(a_\bullet, m_\bullet) \in \mathbb{E}(\mathcal{J})$ and $(X, \xi) \in \mathbf{Coalg}(M, \mathbf{Set})$. If $x \in X(a_0)$ then the maps $(H^{(a_\bullet, m_\bullet)}, \theta^{(a_\bullet, m_\bullet)}) \longrightarrow (X, \xi)$ mapping 1_{a_0} to x correspond to the resolutions of x along (a_\bullet, m_\bullet) .

Proof By the standard Yoneda Lemma, a natural transformation $\alpha : H^{(a_\bullet, m_\bullet)} \longrightarrow X$ amounts to a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in X(a_n)$. It is a map of coalgebras if and only if

$$\begin{array}{ccc} \sum_{n \in \mathbb{N}} \mathbb{A}(a_n, -) & \xrightarrow{\alpha} & X \\ \theta^{(a_\bullet, m_\bullet)} \downarrow & & \downarrow \xi \\ \sum_{n \in \mathbb{N}} M(a_n, -) & \xrightarrow{M \otimes \alpha} & M \otimes X \end{array}$$

commutes, if and only if this diagram commutes when we take 1_{a_n} at the top-left corner for every $n \in \mathbb{N}$, if and only if $\xi(x_n) = m_{n+1} \otimes x_{n+1}$ for all $n \in \mathbb{N}$. A coalgebra map $(H^{(a_\bullet, m_\bullet)}, \theta^{(a_\bullet, m_\bullet)}) \longrightarrow (X, \xi)$ therefore amounts to a sequence $(x_n)_{n \in \mathbb{N}}$ satisfying $\xi(x_n) = m_{n+1} \otimes x_{n+1}$ for all n , that is, a resolution along (a_\bullet, m_\bullet) in (X, ξ) . This establishes the bijection; naturality follows from the naturality in the standard Yoneda Lemma. \square

We have met just one other canonical M -coalgebra: $(\text{ob } \mathcal{J}, \iota)$, constructed in §6. (Recall that M -coalgebras are nondegenerate by definition; $\text{ob } \mathcal{J}$ is nondegenerate whether or not **S** holds.)

Proposition B.4 (Tautological map) *For each complex (a_\bullet, m_\bullet) there is a canonical map of M -coalgebras*

$$\kappa^{(a_\bullet, m_\bullet)} : (H^{(a_\bullet, m_\bullet)}, \theta^{(a_\bullet, m_\bullet)}) \longrightarrow (\text{ob } \mathcal{J}, \iota),$$

satisfying $\kappa^{(a_\bullet, m_\bullet)}(1_{a_0}) = (a_\bullet, m_\bullet)$.

Proof Every complex (a_\bullet, m_\bullet) , regarded as an element of $\text{ob } \mathcal{J}(a_0)$, has a canonical resolution in $\text{ob } \mathcal{J}$. By Lemma B.3, the corresponding map $\kappa^{(a_\bullet, m_\bullet)}$ of coalgebras sends 1_{a_0} to (a_\bullet, m_\bullet) . \square

Proof of Theorem B.1 It remains to prove (c) \implies (a).

Suppose that (\mathbb{A}, M) has a universal solution (J, γ) in **Set**. Then there is a unique map $\beta : (\text{ob } \mathcal{J}, \iota) \longrightarrow (J, \gamma)$ of M -coalgebras. I claim that the natural transformation β can be factorized as

$$\begin{array}{ccc} \text{ob } \mathcal{J} & \xrightarrow{\beta} & J, \\ & \searrow \pi & \nearrow \tilde{\beta} \\ & I & \end{array}$$

where π is the usual projection (§6). Equivalently, for each $a \in \mathbb{A}$ the function $\beta_a : \text{ob } \mathcal{J}(a) \longrightarrow J(a)$ is constant on connected-components of $\mathcal{J}(a)$; equivalently, if $f_\bullet : (a_\bullet, m_\bullet) \longrightarrow (b_\bullet, p_\bullet)$ in $\mathcal{J}(a)$ then $\beta_a(a_\bullet, m_\bullet) = \beta_a(b_\bullet, p_\bullet)$. Indeed, given such an f_\bullet , there are coalgebra maps

$$\begin{array}{ccccc} (H^{(a_\bullet, m_\bullet)}, \theta^{(a_\bullet, m_\bullet)}) & \xrightarrow{\kappa^{(a_\bullet, m_\bullet)}} & (\text{ob } \mathcal{J}, \iota) & \xrightarrow{\beta} & (J, \gamma), \\ f_\bullet^* \uparrow & & \nearrow \kappa^{(b_\bullet, p_\bullet)} & & \\ (H^{(b_\bullet, p_\bullet)}, \theta^{(b_\bullet, p_\bullet)}) & & & & \end{array}$$

and $\beta \circ \kappa^{(a_\bullet, m_\bullet)} \circ f_\bullet^* = \beta \circ \kappa^{(b_\bullet, p_\bullet)}$ by terminality of (J, γ) . (The triangle is not asserted to commute.) But

$$\begin{aligned} \kappa^{(a_\bullet, m_\bullet)} f_\bullet^*(1_a) &= \kappa^{(a_\bullet, m_\bullet)}(1_a \circ f_0) = \kappa^{(a_\bullet, m_\bullet)}(1_a) = (a_\bullet, m_\bullet), \\ \kappa^{(b_\bullet, p_\bullet)}(1_a) &= (b_\bullet, p_\bullet), \end{aligned}$$

so $\beta_a(a_\bullet, m_\bullet) = \beta_a(b_\bullet, p_\bullet)$, as required. This proves the claim. It then follows from Lemma B.2 that (\mathbb{A}, M) satisfies **S**. \square

C Appendix: Realizability

Here we describe the class of topological spaces that can be characterized by some equational system—those that are realizable, in the sense of Definition 2.10. We showed in §8 that all such spaces are compact and metrizable, so the question is: which compact metrizable spaces are realizable? Perhaps surprisingly, the answer turns out to be: all of them (Theorem C.1).

This theorem is less important than it might appear. It characterizes those spaces that admit at least one recursive decomposition, but the same space may admit several such decompositions (Examples 10.1, 10.5, 10.12, 10.13). Compare the result that every nonempty set admits at least one group structure, which does not play an important role in group theory.

It is crucial to this theorem that in the definition of equational system (\mathbb{A}, M) , there may be infinitely many ‘equations’ (objects of \mathbb{A})—even though each individual equation involves only finitely many spaces. In the proof, there is infinite regress: the given space S is decomposed into subspaces S_i ; each S_i is decomposed into subspaces S_{ij} , and so on. Our theorem is one of many stating that the topology of metric spaces can be probed effectively by countable methods: compare, for instance, the fact that a metric space is compact if and only if it is sequentially compact.

There is a similar theorem for discretely realizable spaces. We already know that every such space is compact, metrizable and totally disconnected (Example 6.3); Theorem C.7 states the converse.

The analogous questions for *finite* equational systems are unanswered. Since, up to homeomorphism, there are uncountably many compact metrizable spaces but only countably many finitely realizable spaces, not every compact metrizable space is finitely realizable. It can also be shown that any space realizable by a finite discrete equational system has finite Cantor–Bendixson rank.

Here is our main theorem.

Theorem C.1 (Realizability) *A topological space is realizable if and only if it is compact and metrizable.*

The idea of the proof is as follows. Let S be a compact metrizable space. Cover S by two closed subsets V_1 and V_1' . Then $S = V_1 \cup V_1'$; hence, S is the pushout

$$S = V_1 +_{V_1''} V_1'$$

where $V_1'' = V_1 \cap V_1'$. Next, cover S by a different pair V_2, V_2' of closed subsets and write $V_2'' = V_2 \cap V_2'$: then

$$\begin{aligned} V_1 &= (V_1 \cap V_2) +_{(V_1 \cap V_2'')} (V_1 \cap V_2'), \\ V_1' &= (V_1' \cap V_2) +_{(V_1' \cap V_2'')} (V_1' \cap V_2'), \\ V_1'' &= (V_1'' \cap V_2) +_{(V_1'' \cap V_2'')} (V_1'' \cap V_2'). \end{aligned}$$

Continue in this way to obtain a countable equational system. Compact metrizability of S means that the covers can be chosen to penetrate all of its structure, and the universal solution I is then made up of the space S , the various covering subsets and their intersections, and the inclusions between them.

Given covers \mathcal{W} and \mathcal{V} of a space, \mathcal{W} is said to **refine** \mathcal{V} if for all $W \in \mathcal{W}$ there exists $V \in \mathcal{V}$ such that $W \subseteq V$.

Definition C.2 Let S be a topological space. A **separating sequence** for S is a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ of finite closed covers of S such that

- a. $\mathcal{V}_0 = \{S\}$, and for all $n \in \mathbb{N}$, \mathcal{V}_{n+1} refines \mathcal{V}_n

- b. for all $s, t \in S$ with $s \neq t$, there exists $n \in \mathbb{N}$ such that for all $V \in \mathcal{V}_n$, $\{s, t\} \not\subseteq V$
- c. for all $n \in \mathbb{N}$, for all $V, V' \in \mathcal{V}_n$, we have $V \cap V' \in \mathcal{V}_n$
- d. for all $n \in \mathbb{N}$, for all $V \in \mathcal{V}_n$ and $W \in \mathcal{V}_{n+1}$, we have $V \cap W \in \mathcal{V}_{n+1}$.

The importance of condition (d) is that any element $V \in \mathcal{V}_n$ is covered exactly by elements of \mathcal{V}_{n+1} : indeed,

$$V = V \cap \bigcup_{W \in \mathcal{V}_{n+1}} W = \bigcup_{W \in \mathcal{V}_{n+1}} V \cap W = \bigcup_{X \in \mathcal{V}_{n+1}: X \subseteq V} X. \quad (46)$$

Lemma C.3 *Every compact metrizable space admits a separating sequence.*

Proof Let S be a compact metrizable space. Then S has a countable basis $(U_n)_{n \geq 1}$ of open sets. For each $n \geq 1$, let

$$\mathcal{W}_n = \{\overline{U_n}, S \setminus U_n, \overline{U_n} \cap (S \setminus U_n)\}$$

where $\overline{U_n}$ is the closure of U_n . Then $(\mathcal{W}_n)_{n \geq 1}$ is a sequence of finite closed covers. It satisfies conditions (b) and (c) of Definition C.2, with \mathbb{N} changed to \mathbb{N}^+ : condition (c) is obvious, and for (b), if $s \neq t$ then we may find $n \geq 1$ such that $s \in U_n$ but $t \notin \overline{U_n}$, and then there is no $W \in \mathcal{W}_n$ for which $s, t \in W$. It does not necessarily satisfy (a) or (d); but now define, for each $n \in \mathbb{N}$,

$$\mathcal{V}_n = \{W_1 \cap \cdots \cap W_n \mid W_1 \in \mathcal{W}_1, \dots, W_n \in \mathcal{W}_n\}$$

(understood as $\mathcal{V}_0 = \{S\}$ when $n = 0$). From the properties of $(\mathcal{W}_n)_{n \geq 1}$ stated, it is easily shown that $(\mathcal{V}_n)_{n \in \mathbb{N}}$ is a separating sequence for S . \square

Fix a compact metrizable space S with a separating sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$.

We define an equational system (\mathbb{A}, M) . Recall that a poset can be regarded as a category in which each hom-set has at most one element: there is a map $a' \longrightarrow a$ just when $a' \leq a$. For each $n \geq 0$, let \mathbb{A}_n be the set of nonempty elements of \mathcal{V}_n , ordered by inclusion. Let \mathbb{A} be the coproduct $\sum_{n \in \mathbb{N}} \mathbb{A}_n$, so that an object of \mathbb{A} is a pair (n, V) with $n \in \mathbb{N}$ and $\emptyset \neq V \in \mathcal{V}_n$. Define a module $M : \mathbb{A} \dashrightarrow \mathbb{A}$ by

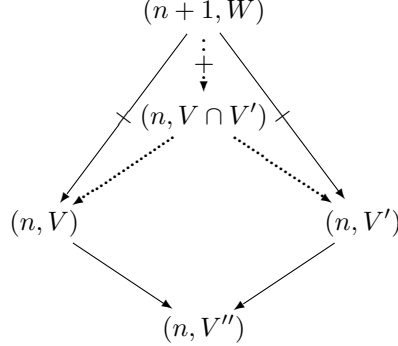
$$M((p, W), (n, V)) = \begin{cases} 1 & \text{if } p = n + 1 \text{ and } W \subseteq V \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, M is also ‘posetal’: there is at most one sector from any object of \mathbb{A} to any other.

Lemma C.4 (\mathbb{A}, M) is an equational system.

Proof Finiteness of M follows from finiteness of each cover \mathcal{V}_n . For nondegeneracy, we verify conditions **ND1** and **ND2**. Condition **ND2** is trivial since \mathbb{A}

is a poset. For **ND1**, take a square of solid arrows



in (\mathbb{A}, M) , so that $W \in \mathcal{V}_{n+1}$, $V, V', V'' \in \mathcal{V}_n$, and $W \subseteq V \cap V'$. Since $W \neq \emptyset$, we have $V \cap V' \neq \emptyset$, that is, $V \cap V' \in \mathbb{A}_n$. Hence the diagram can be filled in with the dotted arrows shown. \square

Define a functor $J : \mathbb{A} \longrightarrow \mathbf{Top}$ on objects by $J(n, V) = V$ (topologized as a subspace of S) and on morphisms by the evident inclusions. We have, for each object (n, V) of \mathbb{A} ,

$$(M \otimes J)(n, V) = \lim_{\rightarrow W \in \mathbb{A}_{n+1}: W \subseteq V} W, \quad (47)$$

where the right-hand side is a colimit over a full subcategory of \mathbb{A}_{n+1} . Hence there is a (continuous) map

$$\psi_{(n, V)} : (M \otimes J)(n, V) \longrightarrow J(n, V) = V \quad (48)$$

whose W -component is the inclusion $W \hookrightarrow V$. This defines a natural transformation $\psi : M \otimes J \longrightarrow J$.

Lemma C.5 $J : \mathbb{A} \longrightarrow \mathbf{Top}$ is a nondegenerate functor, and $\psi : M \otimes J \longrightarrow J$ is an isomorphism.

Proof For the first part, each space $J(n, V) = V$ is a closed subspace of the compact Hausdorff space S , and therefore compact Hausdorff. So it is enough to prove that the underlying **Set**-valued functor of J is nondegenerate. As in the proof of Lemma C.4, condition **ND2** is trivial, and condition **ND1** is an easy check.

For the second part, we have to show that for every object (n, V) of \mathbb{A} , the map $\psi_{(n, V)}$ of (48) is a homeomorphism. Its domain is a finite colimit of compact spaces, hence compact, and its codomain is Hausdorff, so it suffices to show that it is a bijection.

Surjectivity follows immediately from (46).

For injectivity, first note that an element of the colimit (47) is an equivalence class of pairs (W, w) where $w \in W \in \mathcal{V}_{n+1}$ and $W \subseteq V$. The equivalence relation \sim is generated as follows: if $X, W \in \mathcal{V}_{n+1}$ with $x \in X \subseteq W \subseteq V$ then $(X, x) \sim (W, x)$. Writing $[]$ for equivalence class, we have $\psi_{(n, V)}([W, w]) = w$. Now suppose that

$$\psi_{(n, V)}([W, w]) = \psi_{(n, V)}([W', w']).$$

Then $w = w'$, so $w \in W \cap W' \in \mathcal{V}_{n+1}$ with $W \cap W' \subseteq V$. Hence $(W \cap W', w)$ is a pair of the relevant type, and

$$(W, w) \sim (W \cap W', w = w') \sim (W', w'),$$

as required. \square

Proposition C.6 *(J, ψ^{-1}) is the universal solution of (\mathbb{A}, M) in **Top**.*

Proof We verify condition (c) of the Precise Recognition Theorem (9.4). Each space $J(n, V) = V$ is compact, and nonempty by definition of \mathbb{A} , so it only remains to check the main part of the condition.

A complex in (\mathbb{A}, M) is of the form

$$\cdots \xrightarrow{m_2} (n+1, V_{n+1}) \xrightarrow{m_1} (n, V_n)$$

where $V_r \in \mathbb{A}_r$ and $V_n \supseteq V_{n+1} \supseteq \cdots$. We have

$$\psi_{m_1} \circ \cdots \circ \psi_{m_r} = (V_{n+r} \hookrightarrow \cdots \hookrightarrow V_n) = (V_{n+r} \hookrightarrow V_n),$$

so $V_{m_1, \dots, m_r}^J = V_{n+r}$. Hence

$$\bigcap_{r \in \mathbb{N}} V_{m_1, \dots, m_r}^J = \bigcap_{i \geq n} V_i.$$

Suppose that $s, t \in \bigcap_{i \geq n} V_i$. By condition (a) of Definition C.2, there exist $V_{n-1} \in \mathcal{V}_{n-1}, \dots, V_0 \in \mathcal{V}_0$ such that

$$V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_0,$$

and then $s, t \in \bigcap_{i \in \mathbb{N}} V_i$. So by condition (b), $s = t$. \square

Proof of Theorem C.1 Let S be a compact metrizable space and construct (\mathbb{A}, M) and (J, ψ) as above. If S is nonempty then $(0, S)$ is an object of \mathbb{A} , and $J(0, S) = S$. On the other hand, \emptyset is the universal solution of the equational system $(\mathbf{1}, \emptyset)$ (Example 10.1). \square

Theorem C.7 (Discrete realizability) *The following conditions on a topological space S are equivalent:*

- a. S is discretely realizable
- b. S is the limit of some sequence $(\cdots \longrightarrow S_2 \longrightarrow S_1)$ of finite discrete spaces
- c. S is the limit of some countable diagram of finite discrete spaces
- d. S is compact, metrizable, and totally disconnected.

Proof

(a) \implies (b) Let (\mathbb{A}, M) be a discrete equational system. The universal solution is $\text{ob } \mathcal{J}$, and each space $\text{ob } \mathcal{J}(a)$ is the limit of the sequence of the finite discrete spaces $\text{ob } \mathcal{J}_n(a)$ (Example 6.3).

(b) \implies (c) Trivial.

(c) \implies (d) Compact metrizable spaces are the same as compact Hausdorff spaces that are second countable (have a countable basis of open sets). The classes of compact Hausdorff spaces and totally disconnected spaces are closed under all limits, and the class of second countable spaces is closed under countable limits.

(d) \implies (a) For this we adapt the proof of Theorem C.1. We may choose for S a basis $(U_n)_{n \geq 1}$ of open sets that are also closed, by Theorem II.4.2 of [Joh]. The separating sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ constructed in Lemma C.3 then has the property that each cover \mathcal{V}_n is a partition of S . The resulting category \mathbb{A} is therefore discrete, and the result follows. \square

For example, the underlying topological space of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a countable limit of finite discrete spaces, so discretely realizable.

A measure of the power of the realizability theorems is that some classical results of topology [Wil, HY] can be deduced. Proposition 8.5 implies that every realizable space is a topological quotient of a discretely realizable space; thus, every compact metrizable space is a quotient of a totally disconnected compact metrizable space. On the other hand, it can be shown directly that every nonempty discretely realizable space is a retract of the Cantor set. It follows that every totally disconnected compact metrizable space is a subspace of the Cantor set, and that every nonempty compact metrizable space is a quotient of the Cantor set. Finally, it follows from Proposition 10.3 that every totally disconnected compact metrizable space without isolated points is either empty or homeomorphic to the Cantor set. We have thus deduced the classical results characterizing the closed subspaces, quotients and homeomorphism type of the Cantor set. Detailed proofs can be found in [Lei2].

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